Solution

EE103_Fall'18_Final Examination
Dec. 12, 2018 4:00-7:00 p.m.

Name______________________  ID______________________

I. You are allowed to use 2 pages of formulas and tables, but not any concept
descriptions, problem solutions or mathematical derivations.

II. The final course grade will be based on the following weights

   Quiz Average  20% (higher 6 out of 8 )
   Midterm Exam  30%
   Final Exam     50%

This final exam presents 6 problems

[1] 20 points _________
[2] 15 points _________
[3] 20 points _________
[4] 20 points _________
[5] 15 points _________
[6] 10 points _________

Total 100 points _________
[1] (20 points) Given a graph below, plot $x_{\text{even}}(t)$, $x_{\text{odd}}(t)$ on the graph below with proper coordinate values.

\[ x_{\text{even}}(t) = \frac{1}{2} (x(t) + x(-t)) \]

plot $x_{\text{even}}(t)$ here (10 pts)

\[ x_{\text{odd}}(t) = \frac{1}{2} (x(t) - x(-t)) \]

plot $x_{\text{odd}}(t)$ here (10 pts)
[2] (15 points) A linear time-invariant (LTI) system is described below.

\[ x(t) = \delta(t) \quad \text{LTI System} \quad y(t) = h(t) \]

For \( h(t) = e^{-t} u(t-1) \), \( x(t) = 2 \left( u(t) - u(t-2) \right) \), find \( y(t) \) value for \( t=2 \) and a mathematical description for \( y(t) \) for \( t > 3 \).

(5 pts) \( y(2) = \frac{2 \left( e^1 - e^2 \right)}{2} = 2 \left( e^{-1} - e^{-2} \right) \)

(10 pts) \( y(t) = 2 e^t \left( e^{2t} - 1 \right) \)

For \( t > 3 \),

\[ y(t) = \int_{t-2}^{t} e^{x} \, dx = \left. \frac{e^x}{2} \right|_{t-2}^{t} = \frac{e^t}{2} - \frac{e^{t-2}}{2} = \frac{e^t - e^{t-2}}{2} \]

\[ = \frac{e^t}{2} \left( e^{2t} - 1 \right) \]
[3] (20 points) A periodic function \( v(t) \) is plotted below.

(a) (5 pts) Write down a mathematical description of \( v(t) \) by using a generating function \( g(t) \) and a train of delta function \( \delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_\circ) \)

\[
v(t) = g(t) \star \delta_T(t).
\]
In other words, what operation would you use for _______.

Is it addition, multiplication or convolution?

(b) (15 pts) Find Fourier transform \( V(\omega) \) of \( v(t) \).

Step 1 (5 pts) First derive \( G(\omega) \) of \( g(t) \) using Fourier transform.

\[
G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = A \left. \frac{e^{i\omega t}}{i\omega} \right|_{-\frac{T}{2}}^{\frac{T}{2}} = A \frac{1}{i\omega} \left( e^{i\omega \frac{T}{2}} - e^{-i\omega \frac{T}{2}} \right) = AT \frac{\sin \omega \frac{T}{2}}{\omega}
\]

Step 2 (10 pts) Find \( V(\omega) \)

\[
\Delta_T(\omega) = \frac{1}{T} \left[ \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right] = \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0), \quad \omega_0 = \frac{2\pi}{T_\circ}
\]

\[
V(\omega) = G(\omega) \times \Delta_T(\omega) = AT \frac{\sin \omega \frac{T}{2}}{\omega} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = AT \omega_0 \sum_{k=-\infty}^{\infty} \text{sinc} \left( k\omega_0 \frac{T}{2} \right)
\]
[4]. (20 points) For the following filter,

\[ \frac{1}{\sqrt{2}CS} \]

(a). (15 pts) Find \( H(s) = \frac{V_o(s)}{V_i(s)} \)

KCL at node \( \Box \):

\[ \frac{V_i(s)}{R} - \frac{V_0(s)}{R} \frac{(V_o(s) - V_i(s))}{\sqrt{2}CS} = 0 \]

\[ \times R \text{ both sides} \Rightarrow V_i(s) - V_0(s) + V_0(s) - V_0(s) + (V_0(s) - V_0(s)) \sqrt{2} RCS = 0 \]

And \( V_0(s) = V_0(s) + \frac{RC}{\sqrt{2}S} \frac{V_0(s)}{S} \)

\[ \frac{V_0(s)}{S} = V_0(s) \left( 1 + \frac{RC}{\sqrt{2}S} \right) \quad (2) \]

From (1) and (2):

\[ V_i(s) = (2 + \sqrt{2} RCS) V_0(s) - V_0(s) \left( 1 + \sqrt{2} RCS \right) \]

\[ = (2 + \sqrt{2} RCS) V_0(s) \left( 1 + \frac{RC}{\sqrt{2}S} \right) - V_0(s) \left( 1 + \sqrt{2} RCS \right) \]

\[ = V_0(s) \left[ (2 + \sqrt{2} RCS) \left( 1 + \frac{RC}{\sqrt{2}S} \right) - 1 - \sqrt{2} RCS \right] \]

\[ \Rightarrow H(s) = \frac{V_o(s)}{V_i(s)} = \frac{(RC) + \sqrt{2} RCS + 1}{(RC) + \sqrt{2} RCS + 1} \]

(b). (5 pts) Determine the property of this filter. Is it a low-pass or high-pass filter?

Justify your answer.

\[ H(w) = \frac{1}{1 - (RCw)^2 + j \sqrt{2} RCw} \]

\[ |H(w)| = \frac{1}{\sqrt{(1 - (RCw)^2)^2 + 2(RCw)^2}} \rightarrow 0 \text{ as } w \rightarrow \infty \]

\[ = 1 \text{ at } w = 0 \]

\[ \text{Low-pass filter} \]
(5) (15 points) The characteristics of a filter is described below by using locations of poles and zeros (left figure) and also by a plot of its magnitude vs. $\omega$ plot (right figure).

(a) (10 pts) Derive $M_{\text{notch}}(s)$ by using the information above.

$$M_{\text{notch}}(s) = K \frac{(s+j\omega_0)(s-j\omega_0)}{(s+0.1\omega_0+j\omega_0)(s+0.1\omega_0-j\omega_0)} = K \frac{s^2 + \omega_0^2}{(s+0.1\omega_0)^2 + \omega_0^2}$$

$$M_{\text{notch}}(\omega) = K \frac{-\omega^2 + \omega_0^2}{(j\omega + 0.1\omega_0)^2 + \omega_0^2}$$ and $M_{\text{notch}}(0) = K \frac{\omega_0^2}{1-0.1\omega_0^2} = 1$

Thus $K = \frac{1}{1-0.1}$

$$\Rightarrow M_{\text{notch}}(s) = \frac{1}{1-0.1} \frac{s^2 + \omega_0^2}{(s+0.1\omega_0)^2 + \omega_0^2}$$

(b) (5 pts) Find the output $y(t)$ for $t = 0^+$ for input signal $x(t) = \cos(2\omega_0 t)$.

$$X(s) = \frac{s}{s^2 + (2\omega_0)^2}$$

$$Y(s) = M_{\text{notch}}(s) \cdot X(s) = \frac{1}{1-0.1} \frac{s^2 + \omega_0^2}{(s+0.1\omega_0)^2 + \omega_0^2} \cdot \frac{s}{s^2 + (2\omega_0)^2}$$

$$y(t) = \lim_{s \to 0} \frac{sY(s)}{s} = \frac{1}{1-0.1} \lim_{s \to 0} \frac{(s^2 + \omega_0^2)(s^2 + (2\omega_0)^2)}{(s+0.1\omega_0)^2 + \omega_0^2(s^2 + (2\omega_0)^2)}$$

$$= \frac{1}{1-0.1}$$