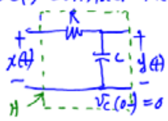
ECE103 Lecture 10, Oct 19, 2018

Application of Fourier series to System Analysis

Let's consider the ~~following~~ RC circuit:




We learned that

$$\delta(t) \xrightarrow[H]{} \frac{1}{RC} e^{-\frac{1}{RC}t}$$

$$u(t) \xrightarrow[H]{} \left(1 - e^{-\frac{1}{RC}t}\right) u(t)$$

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

$$X(s) \xrightarrow{H(s)} Y(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RCs+1} = \left(\frac{1}{RC}\right)\frac{1}{s + \frac{1}{RC}}$$

Let $\alpha = \frac{1}{RC}$

$$H(s) = \frac{\alpha}{s+\alpha}$$

For $s = j\omega$, $H(j\omega) = \frac{\alpha}{\alpha + j\omega} = \frac{\alpha}{\sqrt{\alpha^2+\omega^2}} e^{-j\tan^{-1}\left(\frac{\omega}{\alpha}\right)} = |H(j\omega)| e^{j\angle H(j\omega)}$

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$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t} dt$$

where $C_k$ is a complex number

$C_k = A_k + jB_k$

$\omega_0 = 2\pi f_0 = 2\pi / T_0$

$f_0 = \frac{1}{T_0}$ (period)


$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

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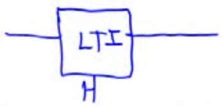


$x_1(t) \longrightarrow y_1(t)$

$a x_1(t) \qquad a y_1(t)$

$x_2(t) \longrightarrow y_2(t)$

$b x_2(t) \qquad b y_2(t)$

$a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t)$

$\sum_{k=1}^{2} C_k e^{jk\omega_0 t} = C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t}$

$\longrightarrow D_1 e^{j(\omega_0 t + \theta_1)}$

$\longrightarrow D_2 e^{j(2\omega_0 t + \theta_2)}$

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$$\mathcal{L}[x(t)] = X(s) \longrightarrow \boxed{H(s)} \longrightarrow \mathcal{L}[y(t)] = Y(s)$$

$$Y(s) = H(s) X(s)$$

test input $Me^{st}$

$\boxed{e^{st}}$

$s = \sigma + j\omega$

If $\sigma = 0$ $s = j\omega$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

$$Y(j\omega) = H(j\omega) X(j\omega)$$

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$$Y(j\omega) = H(j\omega) X(j\omega)$$

$$= |H(j\omega)| e^{j\angle H(j\omega)} \, |X(j\omega)| e^{j\angle X(j\omega)}$$

$$= |H(j\omega)| \cdot |X(j\omega)| \, e^{j(\angle H(j\omega) + \angle X(j\omega))}$$


$$Y(s) = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{RCs+1} = \frac{1}{RC}\frac{1}{s + (1/RC)} \times 1 \; (= \mathcal{L}[\delta(t)])$$

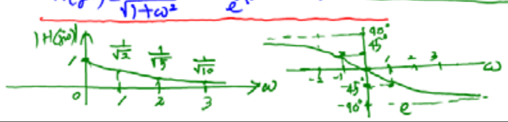
$H(s)$

$s \leftarrow j\omega$

$$\boxed{H(j\omega) = 1 \frac{1}{j\omega + 1}}$$

For simplicity if $\alpha = \frac{1}{RC} = 1$, then

$$H(j\omega) = \frac{1}{\sqrt{1+\omega^2}} \cdot e^{j(-\tan^{-1}\omega)}$$

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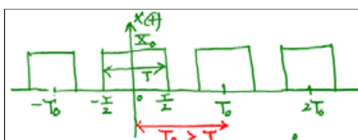



$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \qquad \omega_0 = 2\pi / T_0$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0}\left[\int_{-\frac{T}{2}}^{0} X_0 e^{-jk\omega_0 t} dt + \int_{0}^{\frac{T}{2}} X_0 e^{-jk\omega_0 t} dt\right]$$

$$= \frac{1}{T_0} X_0 \left[\left.\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right|_{-\frac{T}{2}}^{0} + \left.\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right|_{0}^{\frac{T}{2}}\right]$$

$$= \frac{X_0}{T_0}\left[\frac{1}{-jk\omega_0}\left(1 - e^{+jk\omega_0\frac{T}{2}}\right) + \frac{1}{-jk\omega_0}\left(e^{-jk\omega_0\frac{T}{2}} - 1\right)\right]$$

$$c_k = \frac{1}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} X_0\, e^{-jk\omega_0 t}\,dt$$

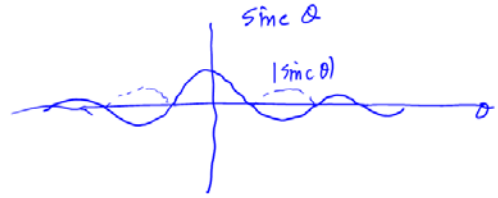
$$= \frac{X_0}{T_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{X_0}{T_0}\frac{1}{-jk\omega_0}\left[e^{-jk\omega_0\frac{T}{2}} - e^{-jk\omega_0\left(-\frac{T}{2}\right)}\right]$$

$$= \;''\; \times(2j)\,(-)\left(\frac{e^{+jk\omega_0\frac{T}{2}} - e^{-jk\omega_0\frac{T}{2}}}{2j}\right)$$

$$= \frac{X_0}{T_0}\frac{\cancel{2j}\,T}{\cancel{jk\omega_0}\,T/2}\sin k\omega_0\frac{T}{2} = X\left(\frac{T}{T_0}\right)\frac{\sin\left(k\omega_0\frac{T}{2}\right)}{\left(k\omega_0\frac{T}{2}\right)}$$

$$\theta_k = k\omega_0\frac{T}{2} \qquad X\left(\frac{T}{T_0}\right)\frac{\sin\theta_k}{\theta_k} = X\left(\frac{T}{T_0}\right)\operatorname{sinc}\theta_k$$




$$= \frac{X_0}{T_0}\frac{1}{-jk\omega_0}\left(\cancel{1} - e^{+jk\omega_0\frac{T}{2}} + e^{-jk\omega_0\frac{T}{2}} - \cancel{1}\right)$$

$$= \frac{X_0}{T_0}\frac{-1}{-jk\omega_0}\left(e^{+jk\omega_0\frac{T}{2}} - e^{-jk\omega_0\frac{T}{2}}\right)$$

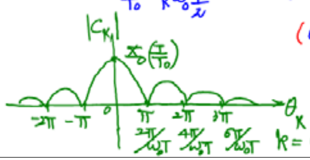
$$= \frac{X_0}{T_0}\frac{1}{k\omega_0}\;\frac{2\left(e^{+jk\omega_0\frac{T}{2}} - e^{-jk\omega_0\frac{T}{2}}\right)}{j2}$$

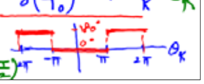
$$= X_0\frac{1}{k\omega_0 T_0\frac{1}{2}}\sin k\omega_0\frac{T}{2}$$

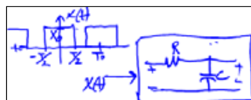
$$= \frac{X_0}{T_0}\frac{T}{k\omega_0\frac{T}{2}}\sin k\omega_0\frac{T}{2} = X_0\left(\frac{T}{T_0}\right)\frac{\sin\theta_k}{\theta_k}$$

$$\left(\theta_k = k\omega_0\frac{T}{2}\right)$$

$$= X_0\left(\frac{T}{T_0}\right)\operatorname{sinc}\theta_k = C_k$$




$$y(t) = \sum C_k^y\, e^{+jk\omega_0 t}$$

$$\frac{1}{RC} = 1$$

$$H(s) = \frac{1}{s+1}$$

$$H(j\omega) = \frac{1}{1+j\omega}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

$$\angle H(j\omega) = -\tan^{-1}\omega$$

$$x(t) = \sum C_k^x\left(= X_0\left(\frac{T}{T_0}\right)\operatorname{sinc}\theta_k\right)e^{+jk\omega_0 t}$$

$$\theta_k = k\,\omega_0\frac{T}{2} = k\,\pi\left(\frac{T}{T_0}\right)$$

where $\omega_0 = 2\pi/T_0$

$$|C_k^y| = |C_k^x|\,|H(jk\omega_0)|$$

$$\angle C_k^y = \angle C_k^x + \angle H(jk\omega_0)$$

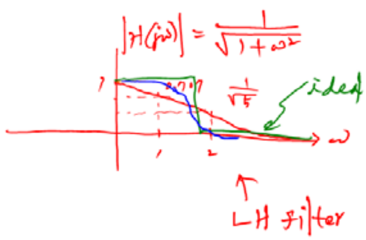
e.g. what is $C_5^y$ ?

$$C_5^y = |C_5^y|\,\angle C_5^y$$

$$|C_5^y| = \left|X_0\left(\frac{T}{T_0}\right)\operatorname{sinc}\left(5\pi\left(\frac{T}{T_0}\right)\right)\right|\,\left|H\left(j\omega\left(=5\omega_0 = 10\pi/T_0\right)\right)\right|$$

$$\angle C_5^y = \angle C_5^x + \angle H(j5\omega_0)$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$




Suppose $T_0 = 10 \quad \omega_0 = 2\pi/T_0 = 0.2\pi$

$T = 2 \quad . \quad T/T_0 = 0.2$

$X_0 = 5$

$$C_k^x = X_0\left(\frac{T}{T_0}\right)\operatorname{sinc}\underbrace{k\overset{2\pi\frac{1}{T_0}}{\omega_0}\frac{T}{2}}_{=k\pi\left(\frac{T}{T_0}\right)}$$

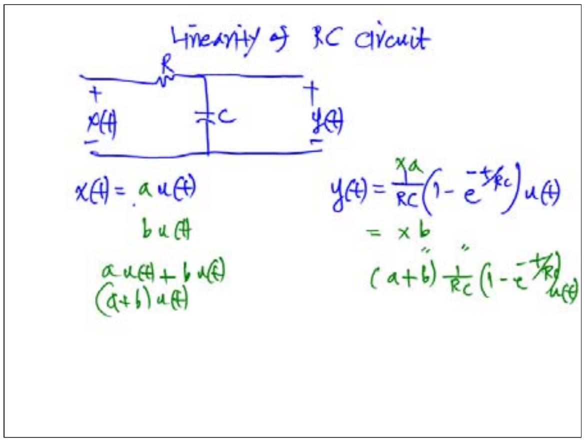
$$C_5^x = 5(0.2)\operatorname{sinc}5\pi(0.2) = 1\operatorname{sinc}\pi = 0$$

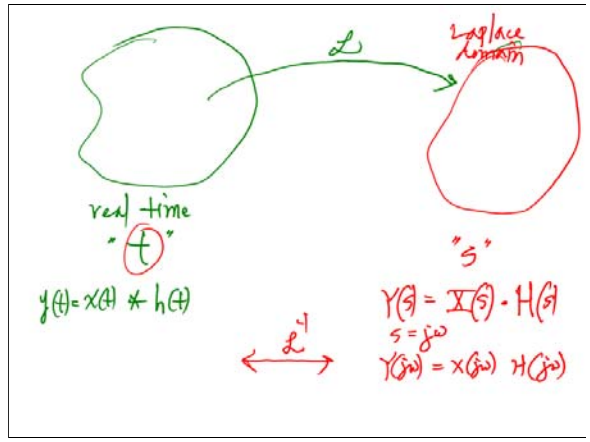
$$C_5^y = |C_5^x|\,|H(j\pi)|\,e^{j\left(\angle\operatorname{sinc}\pi + \angle H(j\pi)\right)}$$

$$= 0 \times \frac{1}{\sqrt{1+\pi^2}}\,e^{j\left(\angle 0^\circ + \angle -\tan^{-1}\pi\right)} = \underline{0}\,e^{j\left(-\tan^{-1}\pi\right)}$$

$$C_6^y = |C_6^x|\,|H(j1.2\pi)|\,e^{j\left(\angle\operatorname{sinc}1.2\pi + \angle -\tan^{-1}1.2\pi\right)}$$

$$= \left(|\operatorname{sinc}(1.2\pi)|\frac{1}{\sqrt{1+(1.2\pi)^2}}\right)e^{j\left(180^\circ - \tan^{-1}1.2\pi\right)}$$


Linearity of RC circuit
R
C
+
x(t)
−
+
y(t)
−
x(t) = a u(t)
b u(t)
a u(t) + b u(t)
(a+b) u(t)
y(t) = xa/RC (1 − e^(−t/RC)) u(t)
= x b
(a+b) 1/RC (1 − e^(−t/RC)) u(t)


L
Laplace domain
real time
"t"
y(t) = x(t) * h(t)
L⁻¹
"s"
Y(s) = X(s) · H(s)
s = jω
Y(jω) = X(jω) H(jω)