ECE103 Lecture 11, Oct 22, 2018

Let's review on systems concept

Input → LTI System (H) → output

$$\delta(t) \xrightarrow{H} h(t) = H(\delta(t))$$

$$x(t) \xrightarrow{H} y(t) = x(t) * h(t) = h(t) * x(t)$$

when $x(t)$ is periodic with period T_0 its fundamental frequency is $f_0 = \frac{1}{T_0}$ and angular frequency is $2\pi f_0 = 2\pi / T_0 = \omega_0$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} 2|c_k| \cos(k\omega_0 t + \theta_k)$$

$$\sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$$

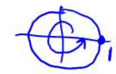
$$= \cdots + c_{-1} e^{-j\omega_0 t} + c_0 + c_1 e^{+j\omega_0 t} + \cdots$$

$$|c_1| e^{+j(-\theta_1)} e^{-j\omega_0 t} + c_0 + |c_1| e^{+j\theta_1} e^{+j\omega_0 t}$$

$$|c_1| \left[e^{-j(\omega_0 t + \theta_1)} + e^{+j(\omega_0 t + \theta_1)} \right]$$

$$|c_1| \cdot 2\cos(\omega_0 t + \theta_1)$$

$e^{\pm j\omega k} = 1$ $\quad e^{0} \quad e^{\pm j 2\pi k} \quad e^{\pm j\pi k}$



Posted are

- ✓ HW #4
- ✓ HW #3 solutions
- ✓ Midterm Exam (Nov, 2017)

Quiz 2 (71 students)

Average = 4.37

σ = 2.59

For $x(t)$ periodic

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2|c_k| \cos(k\omega_0 t + \theta_k)$$

where $c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt = a_k + j b_k$ (b_k can be negative)

$c_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$ = DC value of $x(t)$

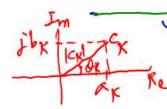
We can also express $x(t)$ as

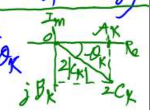
$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[2|c_k| \cos\theta_k \cos k\omega_0 t - 2|c_k| \sin\theta_k \sin k\omega_0 t \right]$$

$$\Rightarrow x(t) = c_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$$

where $A_k = 2|c_k| \cos\theta_k$

$B_k = -2|c_k| \sin\theta_k$





$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$c_k e^{jk\omega_0 t}$ → H LTI → $c_k^y e^{jk\omega_0 t}$

no change in frequency when H is LTI.

$H(s) \leftarrow H(j\omega = jk\omega_0)$

$$c_k^y = |c_k| \cdot |H(jk\omega_0)| \, e^{j(\theta_k + \angle H(jk\omega_0))}$$

holds for all k

$$x(t) = \sum c_k^x e^{jk\omega_0 t} \xrightarrow{H(j\omega)} y(t) = \sum c_k^y e^{jk\omega_0 t}$$

$$= c_0^x + \sum_{k=1}^{\infty} 2|c_k^x| \cos(k\omega_0 t + \theta_k^x) \qquad = c_0^y + \sum_{k=1}^{\infty} 2|c_k^y| \times \cos(k\omega_0 t + \theta_k^y)$$

$$\Rightarrow |c_k^y| = |c_k^x| \, |H(jk\omega_0)|, \quad \theta_k^y = \theta_k^x + \angle H(jk\omega_0)$$

$C_1^x = 2e^{j3t}$ → $y(t) = C^y_{k=1} e^{j3t}$

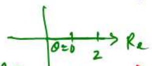
$H(j\omega) = \frac{1}{j\omega}$

$\omega_0 = 3$

$H(j3) = \frac{1}{j3}$

$x(t) = \sum C_k e^{jk\omega_0 t}$

$C_1^x = 2 = |2| e^{j0°}$

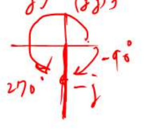


$C_1^y = |C_1^x|\,|H(j3)|\, e^{j\left(0° + \angle\frac{1}{j3}\right)}$

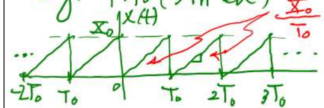
$= 2\left(\frac{1}{3}\right) e^{j(0° + (-90°))}$

$= \frac{2}{3} e^{+j(-90°)}$

$\frac{1}{j3} = \frac{j}{(jj)3} =$



e.g. 4.10 (5th ed.)



period T_0, $\omega_0 = 2\pi\left(\frac{1}{T_0}\right)$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \frac{X_0}{T_0} t\, e^{-jk\omega_0 t} dt;$$

let's choose the interval $[0\ T_0]$

$$= \frac{1}{T_0}\frac{X_0}{T_0}\int_0^{T_0} t\, d\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right)$$

$k \neq 0$

For $k=0$

$$C_0 = \frac{1}{T_0}\frac{X_0}{T_0}\int_0^{T_0} t\, dt = \frac{X_0}{(T_0)^2}\frac{1}{2}T_0^2 = \frac{X_0}{2}$$

$$= \frac{1}{T_0}\frac{X_0}{T_0}\left[t\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right) - \int \frac{e^{-jk\omega_0 t}}{-jk\omega_0}\, dt\right]_0^{T_0}$$

$$= \frac{1}{T_0}\frac{X_0}{T_0}\left[t\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right) - \frac{e^{-jk\omega_0 t}}{(-jk\omega_0)^2}\right]_0^{T_0}$$

$$= \frac{1}{T_0}\frac{X_0}{T_0}\left[T_0\left(\frac{e^{-jk\omega_0 T_0}}{-jk\omega_0}\right) - 0 - \frac{e^{-jk\omega_0 T_0}}{(-jk\omega_0)^2} + \frac{1}{(-jk\omega_0)^2}\right]$$

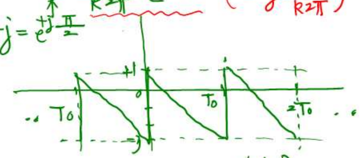
$e^{-jk\omega_0 T_0} = e^{-jk2\pi} = 1$

$$= \frac{1}{T_0}\frac{X_0}{T_0}\left[T_0\left(\frac{1}{-jk\omega_0}\right) - 0 - \cancel{\frac{1}{(-jk\omega_0)^2}} + \cancel{\frac{1}{(-jk\omega_0)^2}}\right]$$

$$= \frac{X_0}{T_0}\frac{1}{-jk\omega_0} = j\frac{X_0}{T_0}\frac{1}{k\omega_0} = j\frac{X_0}{k2\pi}$$

$$= \frac{X_0}{k2\pi} e^{+j\frac{\pi}{2}} \quad \left(= j\frac{X_0}{k2\pi}\right)$$

$+j = e^{+j\frac{\pi}{2}}$



$$y(t) = (-)x(t) \times \left(\frac{4}{X_0}\right) + 1 = ax(t) + b$$

amplitude transformation

$$\sum C_k^y e^{+jk\omega_0 t} = a\sum C_k^x e^{+jk\omega_0 t} + b$$

$\Rightarrow \boxed{C_k^y = aC_k^x}$

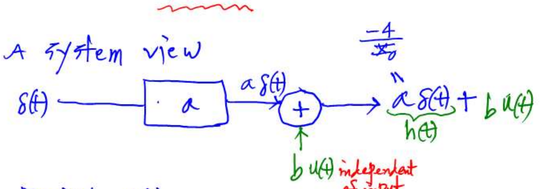
$$C_k^y = aC_k^x + b = -\frac{4}{X_0}C_k^x + 1$$

$$\Rightarrow C_0^y = -\frac{4}{X_0}C_0^x + 1 = -\frac{4}{X_0}\frac{X_0}{2} + 1 = -1$$

$$C_k^y = -\frac{4}{X_0}\left(j\frac{X_0}{k2\pi}\right) = \frac{-2j}{k\pi} \quad \left(\text{where } -j = e^{-j\frac{\pi}{2}}\right)$$

$$= \frac{2}{k\pi} e^{-j\frac{\pi}{2}}$$

A system view



For input $x(t)$, $y(t) = x(t) * h(t) + bu(t)$

$= \int x(\tau)\, a\delta(t-\tau)\, d\tau + bu(t)$

$= ax(t) + bu(t) = H(x(t)) + bu(t)$

$$H(s) = \frac{Y(s)}{X(s)} = a$$

$H(j\omega) = a$, independent of ω

$= |a|\, e^{j\angle a} = \frac{4}{X_0} e^{j\pi}$

$a = -\frac{4}{X_0}$

note that

$y(t) = ax(t) + b$

$Y(s) = aX(s) + \frac{b}{s}$

$\frac{Y(s)}{X(s)} = ?$ when $b = 0$, $H(s) = a$

Time transformation

$x(t)$, $y(t)=x(-t)$, $y(t)=\sum_{k=-\infty}^{\infty} C_k^x e^{+jk\omega_0(-t)}$

$= \sum C_k^x e^{+j(-k)\omega_0 t}$

$= \sum (C_{-k}^x)^* e^{+j(-k\omega_0 t)} = \sum (C_k^x)^* e^{+jk\omega_0 t}$

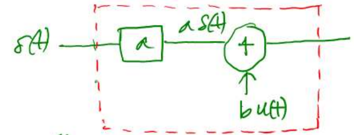
$= \sum C_k^y e^{jk\omega_0 t}$

TABLE 4.8 Amplitude and Time Transformations

Amplitude	$y(t) = Ax(t) + B$ $C_{0y} = AC_{0x} + B$ $C_{ky} = AC_{kx}, k \neq 0$
Time	$\tau = -t \Rightarrow C_{ky} = C_{kx}^*$ $\tau = t - t_0 \Rightarrow C_{ky} = C_{kx}e^{-jk\omega_0 t_0}$

If $y(t) = x(t-t_0) = \sum C_k^x e^{+jk\omega_0(t-t_0)} = \sum \underbrace{C_k^x e^{-jk\omega_0 t_0}} e^{+jk\omega_0 t}$

$= \sum \underbrace{C_k^y} e^{+jk\omega_0 t}$



$a\delta(t) + \underbrace{b\,(t)}_{\text{independent of input}}$

$x_1(t) \longrightarrow ax_1(t) + bu(t)$

$x_2(t) \longrightarrow ax_2(t) + bu(t)$

$k_1x_1(t) + k_2x_2(t)$ $\quad k_1ax_1(t) + bu(t)$
$\quad + k_2ax_2(t) + bu(t)$
$\quad \neq k_1y_1(t) + k_2y_2(t)$

$\boxed{\text{system}}$ is not Linear !

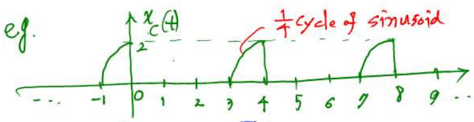
TABLE 4.9 Key Equations of Chapter 4

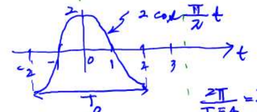
Equation Title	Equation Number	Equation
Exponential form of Fourier series	(4.11)	$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$; $C_k = C_{-k}^*$
Combined trigonometric form of Fourier series	(4.13)	$x(t) = C_0 + \sum_{k=1}^{\infty} 2\lvert C_k\rvert\cos(k\omega_0 t + \theta_k)$
Trigonometric form of Fourier series	(4.17)	$x(t) = A_0 + \sum_{k=1}^{\infty}[A_k\cos k\omega_0 t + B_k \sin k\omega_0 t]$, $A_k - jB_k = 2C_k$, $A_0 = C_0$
Relation of different forms of Fourier coefficients	(4.18)	$2C_k = A_k - jB_k$; $C_k = \lvert C_k\rvert e^{j\theta_k}$; $C_0 = A_0$
Fourier series coefficients formula	(4.23)	$C_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$
Sinc function	(4.29)	$\text{sinc}\, x = \frac{\sin x}{x}$
Steady-state output expressed as Fourier series	(4.38)	$x(t) = \sum_{k=-\infty}^{\infty} C_{kx}e^{jk\omega_0 t} \rightarrow y_{ss}(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0)C_{kx}e^{jk\omega_0 t}$
Fourier coefficients of output signal	(4.39)	$y_{ss}(t) = \sum_{k=-\infty}^{\infty} C_{ky}e^{jk\omega_0 t}$, $C_{ky} = H(jk\omega_0)C_{kx}$

SS = Steady state

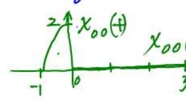
when t very large

e.g.





$\frac{2\pi}{T_0=4} = \frac{\pi}{2} = \omega_0$



$x_{00}(t) = 2\cos\frac{\pi}{2}t\,[u(t+1) - u(t-3)]$

$x_{01}(t) = x_{00}(t-4)$

$x_{0k}(t) = x_{00}(t-4k)$

$x_c(t) = \sum_{k=-\infty}^{\infty} x_{0k}(t) = \underline{\sum_{k=-\infty}^{\infty} 2\cos\frac{\pi}{2}(t-4k)\,[u(t-4k+1) - u(t-4k-3)]}$

$C_k = \frac{1}{T_0}\int_{T_0} x_c(t)\, e^{-jk\omega_0 t}\, dt$

where $T_0 = 4$, $\omega_0 = \frac{\pi}{2}$

$C_k = \frac{1}{4}\int_{-1}^{0} 2\cos\frac{\pi}{2}t\; e^{-jk\frac{\pi}{2}t}\, dt$

$= \frac{1}{4}\int_{-1}^{0} \cancel{2}\, \frac{e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}}{\cancel{2}}\, e^{-jk\frac{\pi}{2}t}\, dt$

$= \frac{1}{4}\int_{-1}^{0} \left[e^{+j\frac{\pi}{2}(1-k)t} + e^{-j\frac{\pi}{2}(1+k)t}\right] dt$

For $k=0$ $\quad C_0 = \frac{1}{4}\int_{-1}^{0} 2\cos\frac{\pi}{2}t\, dt$

$= \frac{1}{4}\left[\frac{2}{\pi}\, 2\sin\frac{\pi}{2}t\right]_{-1}^{0} = \frac{1}{\pi} = C_0$

$k=1$ $\quad C_1 = \frac{1}{4}\int_{-1}^{0} (1 + e^{-j\pi t})\, dt = \frac{1}{4}\left[t - \frac{1}{j\pi}e^{-j\pi t}\right]_{-1}^{0}$

$= \frac{1}{4}\left(0 - (-1) - \frac{1}{j\pi}\left(e^{0} - e^{-j\pi(-1)}\right)\right)$

$= \frac{1}{4}\left(1 - \frac{1}{j\pi}(1-(-1))\right) = \frac{1}{4}\left(1 - \frac{2}{j\pi}\right)$

$= \frac{1}{4}\left(1 + j\frac{2}{\pi}\right) = C_1$

For $k \geq 2$,

$= \frac{1}{4}\int_{-1}^{0}\left(e^{j\frac{\pi}{2}(1-k)t} + e^{-j\frac{\pi}{2}(1+k)t}\right) dt$

$= \frac{1}{4}\left\{\frac{e^{j\frac{\pi}{2}(1-k)t}}{j\frac{\pi}{2}(1-k)} + \frac{e^{-j\frac{\pi}{2}(1+k)t}}{-j\frac{\pi}{2}(1+k)}\right\}_{-1}^{0}$

$= \frac{1}{j\,2\pi(1-k^2)}\left((1+k)e^{j\frac{\pi}{2}(1-k)t} - (1-k)e^{-j\frac{\pi}{2}(1+k)t}\right)_{-1}^{0}$

$$= \frac{j}{2\pi(k^2-1)}\left[2k - k\left(e^{j\frac{\pi}{2}(k-1)} + e^{j\frac{\pi}{2}(k+1)}\right)\right]$$

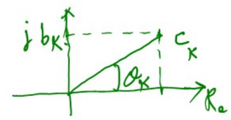
$$\underbrace{e^{j\frac{\pi}{2}k}(-j) + e^{j\frac{\pi}{2}k}(+j)}_{=0}$$

$$= j\frac{k}{\pi(k^2-1)} = c_k, \quad k \geq 2$$

$k=2$, $c_2 = j\frac{2}{\pi\cdot 3} = j\frac{2}{3\pi}$

$k=3$, $c_3 = j\frac{3}{\pi\cdot 8} = j\frac{3}{8\pi}$

$k=4$, $c_4 = j\frac{4}{\pi\cdot 15} = j\frac{4}{15\pi}$



$$c_k = |c_k| e^{j\theta_k}$$

$$= \underbrace{|c_k|\cos\theta_k}_{a_k} + j\underbrace{|c_k|\sin\theta_k}_{b_k}$$

$$\theta_k = \tan^{-1}\frac{b_k}{a_k}$$

$\frac{b_k}{a_k}$	$\theta_k = \tan^{-1}\frac{b_k}{a_k}$
1	45°
2	63.435°
3	71.565°
4	75.964°