ECE 103 Lecture 12 Oct. 24, 2018

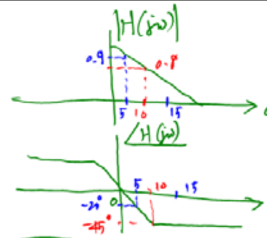
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$H(s),\ H(j\omega)$

$$x(t)=\sum C_k^x e^{jk\omega_0 t} \xrightarrow{H(j\omega)\big|_{\omega=k\omega_0}} y(t)=\sum C_k^y e^{jk\omega_0 t}$$

$$= C_0^x + \sum_{k=1}^{\infty} 2|C_k^x|\cos(k\omega_0 t+\theta_k^x) \qquad = C_0^y + \sum_{k=1}^{\infty} 2|C_k^y| \times \cos(k\omega_0 t+\theta_k^y)$$

$$\Rightarrow |C_k^y| = |C_k^x|\,|H(jk\omega_0)|,\quad \theta_k^y = \theta_k^x + \angle H(jk\omega_0)$$



$H(j\omega) = |H(j\omega)|\,\angle H(j\omega)$

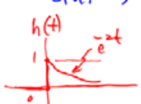
$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

e.g. $\cos(10t)$ ($\omega=10$) $\longrightarrow y = ?$

$y = 0.8\cos(10t-45°)$

$+\ 2\sin 5t \longrightarrow\ y = 2\times 0.9\sin(5t-20°)$

Quiz 3 $\quad h(t) = e^{-2t}u(t)$



$$H(s) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt = \int_0^{\infty} e^{-(s+2)t}\,dt$$

$$= \frac{e^{-(s+2)t}}{-(s+2)}\Bigg|_0^{\infty} = 0 + \frac{1}{s+2} = H(s)$$

R, $\frac{1}{cs} \leftarrow \frac{1}{j\omega c}$ $\qquad H(j\omega) = \frac{1}{2+j\omega}$

$$\cos 8t \longrightarrow \boxed{H(j\omega)} \longrightarrow$$

$\cos 8t = \frac{e^{+j8t}+e^{-j8t}}{2}$

$C_k^x = \frac{1}{2},\ (C_{-k}^x = \frac{1}{2})$

$C_k^y = C_k^x\, H(j8)$

$= \frac{1}{2}\cdot\frac{1}{2+j8}$

$= \frac{1}{2}\,\frac{1}{\sqrt{68}}\,e^{-j\tan^{-1}4}$

$= \frac{1}{4\sqrt{17}}\,e^{-j75.964°}$

$\frac{1}{2\sqrt{17}}$

$$\cos 8t \xrightarrow{H(j\omega)} \frac{2}{4\sqrt{17}}\cos(8t-75.964°)$$

$$C_{-k}^y = \frac{1}{2}\,\frac{1}{2-j8} = \frac{1}{4\sqrt{17}}\,e^{+j75.964°}$$

$$y(t) = C_k^y e^{+j8t} + C_{-k}^y e^{-j8t}$$

$$= \left(\frac{1}{4\sqrt{17}}e^{-j75.964°}\right)e^{+j8t} + \left(\frac{1}{4\sqrt{17}}e^{+j75.964°}\right)e^{-j8t}$$

$$= \frac{1}{4\sqrt{17}}\Big(\underbrace{e^{+j(8t-75.964°)} + e^{-j(8t-75.964°)}}_{=2\cos(8t-75.964°)}\Big)$$

$$= \frac{1}{2\sqrt{17}}\cos(8t-75.964°)$$

Let's also consider $\sin 2t$ as a part of $x(t)$

$$\sin 2t \longrightarrow \boxed{\frac{1}{2+j\omega}} \longrightarrow$$

$\omega = 2$

$$\sin 2t = \frac{e^{j2t}-e^{-j2t}}{2j}$$

$C_k^x = \frac{1}{2j}, \quad H(j2) = \frac{1}{2+j2}$

$$C_k^y = \frac{1}{2j}\,\frac{1}{2+j2} = \frac{1}{2}\,\frac{1}{2\sqrt{2}}\,e^{j(-90°-45°)}$$

$$= \frac{1}{4\sqrt{2}}\,e^{-j135°}$$

$$\sin 2t \xrightarrow{\quad H \quad} \frac{2}{4\sqrt{2}}\sin(2t-135°)$$

$$= \frac{1}{2\sqrt{2}}\sin\big((2t-45°)-90°\big)$$

$$= -\frac{1}{2\sqrt{2}}\cos(2t-45°)$$

$$\sin 2t = \frac{e^{+j2t} - e^{-j2t}}{2j}$$

$$= \begin{cases} \frac{e^{+j2t}}{2j} \longrightarrow \boxed{H(j2)} \longrightarrow y_1 \\ \frac{-e^{-j2t}}{2j} \longrightarrow \boxed{H(-j2)} \longrightarrow y_2 \end{cases}$$

$y_1 + y_2$

$\Downarrow$

$2\times|C_k|$

(Example)

$$\longrightarrow \boxed{\int} \longrightarrow$$

$$h(t) = u(t) \longrightarrow H(s) = \frac{1}{s}$$

$$H(j\omega) = \frac{1}{j\omega}$$

$$|H(j\omega)| = \frac{1}{\omega}, \quad \angle H(j\omega) = -90^\circ$$

$$\sin 3t \longrightarrow \boxed{H} \longrightarrow \; ? \quad y(t) = x(t)\left[\frac{1}{\omega} e^{-j(90^\circ)}\right]$$

$$\underset{\omega=3}{=} \sin 3t \;\; \frac{1}{3} e^{-j90^\circ}$$

$$= \frac{1}{3}\sin(3t - 90^\circ)$$

$$= \frac{1}{3}[\sin 3t \cos(-90^\circ) - \cos 3t \underbrace{\sin 90^\circ}_{=1}]$$

$$= -\frac{1}{3}\cos 3t$$

$$\int \sin 3t \, dt = -\frac{1}{3}\cos 3t \quad \checkmark$$

[Example]

$$\longrightarrow \boxed{d/dt} \longrightarrow$$

$$\omega \; j\omega \; \frac{\pi}{2}$$

$$h(t) = \frac{d\delta(t)}{dt}$$

$$H(s) = s, \quad H(j\omega) = j\omega = \omega e^{+j90^\circ}$$

$$\sin 3t \longrightarrow \boxed{H} \longrightarrow \sin 3t \; \omega e^{j90^\circ}$$

$$\underset{\omega=3}{\rightarrow} 3\sin(3t + 90^\circ)$$

$$= 3(\sin 3t \underbrace{\cos 90^\circ}_{=0} + \cos 3t \underbrace{\sin 90^\circ}_{=1})$$

$$= 3\cos 3t$$

$$\frac{d}{dt}(\sin 3t) = 3\cos 3t \quad \checkmark$$

$$j1 \quad 1e^{j\frac{\pi}{2}} \qquad j1 \; 90^\circ \qquad e^{-j1} \quad -j1 \quad -90^\circ \qquad 1e^{-j\frac{\pi}{2}}$$

$$h(t) \longrightarrow H(s)$$

$$\int_{\cancel{-\infty}\,0}^{\infty} h(t) e^{-st} dt$$

$h(t)$ = impulse response with $\delta(t)$ at $t=0$
then causal, only for $t \geq 0$

e.g. 4.10 (5th ed.) $\frac{X_0}{T_0}$

period T_0, $\omega_0 = 2\pi\left(\frac{1}{T_0}\right)$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

For $k=0$

$$c_0 = \frac{1}{T_0}\frac{X_0}{T_0}\int_0^{T_0} t\,dt = \frac{X_0}{(T_0)^2}\frac{1}{2}T_0^2 = \frac{X_0}{2}$$

let's choose the interval $[0 \; T_0]$

$$= \frac{1}{T_0}\int_0^{T_0} \frac{X_0}{T_0} t \, e^{-jk\omega_0 t} dt;$$

$$\underset{k \neq 0}{=} \frac{1}{T_0}\frac{X_0}{T_0}\int_0^{T_0} t \, d\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right)$$

$$= \frac{1}{T_0}\frac{X_0}{T_0}\left[t\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right) - \int \frac{e^{-jk\omega_0 t}}{-jk\omega_0} dt\right]_0^{T_0}$$

$$= \frac{1}{T_0}\frac{X_0}{T_0}\left[t\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right) - \frac{e^{-jk\omega_0 t}}{(-jk\omega_0)^2}\right]_0^{T_0}$$

$$= \frac{1}{T_0}\frac{X_0}{T_0}\left[T_0\left(\frac{e^{-jk\omega_0 T_0}}{-jk\omega_0}\right) - 0 - \frac{e^{-jk\omega_0 T_0}}{(-jk\omega_0)^2} + \frac{1}{(-jk\omega_0)^2}\right]$$

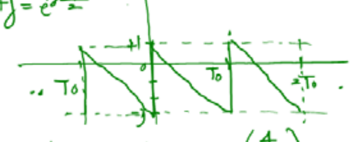
$$e^{-jk\omega_0 T_0} \rightarrow \frac{1}{T_0}\frac{X_0}{T_0}\left[T_0\left(\frac{1}{-jk\omega_0}\right) - 0 - \cancel{\frac{1}{(-jk\omega_0)^2}} + \cancel{\frac{1}{(-jk\omega_0)^2}}\right]$$

$$= e^{-jk2\pi} = 1$$

$$= \frac{X_0}{T_0}\frac{1}{-jk\omega_0} = j\frac{X_0}{T_0}\frac{1}{k\omega_0} = j\frac{X_0}{k2\pi}$$

$$= \frac{X_0}{k2\pi} e^{+j\frac{\pi}{2}} \quad \left(= j\frac{X_0}{k2\pi}\right)$$

$$+j = e^{+j\frac{\pi}{2}}$$



$$y(t) = (-)x(t) \times \left(\frac{4}{X_0}\right) + 1 \quad = a x(t) + b$$

amplitude transformation

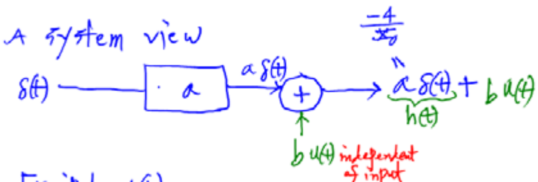
$$\sum c_k^y e^{+jk\omega_0 t} = a\sum c_k^x e^{+jk\omega_0 t} + b$$

$$\Rightarrow c_k^y = a c_k^x$$

$$C_k^y = -\frac{4}{x_0}\left(j\frac{x_0}{k2\pi}\right) = \frac{-2j}{k\pi} \quad \left(\text{where } -j = e^{-j\frac{\pi}{2}}\right)$$

$$= \frac{2}{k\pi} e^{-j\frac{\pi}{2}}$$

A system view



For input $x(t)$, $y(t) = x(t) * h(t) + b\,u(t)$

$$= \int x(\tau)\, a\,\delta(t-\tau)\,d\tau + b\,u(t)$$

$$= a x(t) + b\,u(t) = H(x(t)) + b\,u(t)$$

$$H(s) = \frac{Y(s)}{X(s)} = a$$

$H(j\omega) = a$, independent of ω

$$= |a|\, e^{j\angle a} = \frac{4}{x_0} e^{j\pi}$$

$$a = -\frac{4}{x_0}$$

note that

$$y(t) = a x(t) + b$$

$$Y(s) = a X(s) + \frac{b}{s}$$

$$\frac{Y(s)}{X(s)} = ? \quad \text{when } b = 0,\ H(s) = a$$

Time transformation

$x(t)$, $y(t) = x(-t)$, $y(t) = \sum_{k=-\infty}^{\infty} C_k^x e^{+jk\omega_0(-t)}$

$$= \sum C_k^x e^{+j(-k)\omega_0 t}$$

$$= \sum (C_{-k}^x)^* e^{+j(-k\omega_0 t)} = \sum \underbrace{(C_k^x)^*}_{= C_k^y} e^{+jk\omega_0 t}$$

TABLE 4.8 Amplitude and Time Transformations

Amplitude	$y(t) = Ax(t) + B$
	$C_{0y} = AC_{0x} + B$
	$C_{ky} = AC_{kx}, k \neq 0$
Time	$\tau = -t \Rightarrow C_{ky} = C^*_{kx}$
	$\tau = t - t_0 \Rightarrow C_{ky} = C_{kx} e^{-jk\omega_0 t_0}$

If $y(t) = x(t - t_0) = \sum C_k^x e^{+jk\omega_0(t-t_0)} = \sum \underbrace{C_k^x e^{-jk\omega_0 t_0}}_{} e^{+jk\omega_0 t}$

$$= \sum C_k^y e^{+jk\omega_0 t}$$

$x_1(t) \longrightarrow a x_1(t) + b\,u(t)$

$x_2(t) \longrightarrow a x_2(t) + b\,u(t)$

$k_1 x_1(t) + k_2 x_2(t)$ $\quad$ $k_1 a x_1(t) + b\,u(t) + k_2 a x_2(t) + b\,u(t)$

$$\neq k_1 y_1(t) + k_2 y_2(t)$$

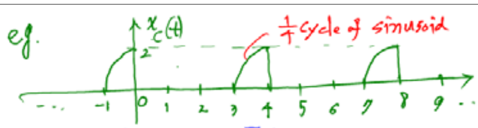
system is not Linear !

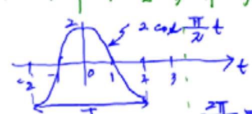
TABLE 4.9 Key Equations of Chapter 4

Equation Title	Equation Number	Equation
Exponential form of Fourier series	(4.11)	$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t};\ C_k = C^*_{-k}$
Combined trigonometric form of Fourier series	(4.13)	$x(t) = C_0 + \sum_{k=1}^{\infty} 2\lvert C_k\rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric form of Fourier series	(4.17)	$x(t) = A_0 + \sum_{k=1}^{\infty}[A_k \cos k\omega_0 t + B_k \sin k\omega_0 t]$, $A_k - jB_k = 2C_k, A_0 = C_0$
Relation of different forms of Fourier coefficients	(4.18)	$2C_k = A_k - jB_k;\ C_k = \lvert C_k\rvert e^{j\theta_k};\ C_0 = A_0$
Fourier series coefficients formula	(4.23)	$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\,dt$
Sinc function	(4.29)	$\text{sinc}\ x = \frac{\sin x}{x}$
Steady-state output expressed as Fourier series	(4.38)	$x(t) = \sum_{k=-\infty}^{\infty} C_{kx} e^{jk\omega_0 t} \rightarrow y_{ss}(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) C_{kx} e^{jk\omega_0 t}$
Fourier coefficients of output signal	(4.39)	$y_{ss}(t) = \sum_{k=-\infty}^{\infty} C_{ky} e^{jk\omega_0 t},\ C_{ky} = H(jk\omega_0) C_{kx}$

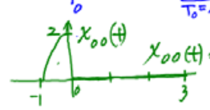
SS = Steady state when t very large

eg.





$$\frac{2\pi}{T_0 = 4} = \frac{\pi}{2} = \omega_0$$



$$x_{00}(t) = 2\cos\frac{\pi}{2}t\,[u(t+1) - u(t-3)]$$

$$x_{0k}(t) = x_{00}(t - 4k)$$

$$x_c(t) = \sum_{k=-\infty}^{\infty} x_{0k}(t) = \sum_{k=-\infty}^{\infty} 2\cos\frac{\pi}{2}(t-4k)\,[u(t-4k+1) - u(t-4k-3)]$$

$$c_k = \frac{1}{T_0}\int_{T_0} x_c(t)\, e^{-jk\omega_0 t}\, dt$$

where $T_0 = 4$, $\omega_0 = \frac{\pi}{2}$

$$c_k = \frac{1}{4}\int_{-1}^{0} 2\cos\frac{\pi}{2}t \; e^{-jk\frac{\pi}{2}t}\, dt$$

$$= \frac{1}{4}\int_{-1}^{0} \cancel{2}\, \frac{e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}}{\cancel{2}}\, e^{-jk\frac{\pi}{2}t}\, dt$$

$$= \frac{1}{4}\int_{-1}^{0}\left[e^{+j\frac{\pi}{2}(1-k)t} + e^{-j\frac{\pi}{2}(1+k)t}\right] dt$$

For $k=0$ $\quad c_0 = \frac{1}{4}\int_{-1}^{0} 2\cos\frac{\pi}{2}t\, dt$

$$= \frac{1}{4}\left[\frac{2}{\pi}\, 2\sin\frac{\pi}{2}t\right]_{-1}^{0} = \frac{1}{\pi} = C_0$$

$k=1$ $\quad c_1 = \frac{1}{4}\int_{-1}^{0}\left(1 + e^{-j\pi t}\right) dt = \frac{1}{4}\left[t - \frac{1}{j\pi}e^{-j\pi t}\right]_{-1}^{0}$

$$= \frac{1}{4}\left(0-(-1) - \frac{1}{j\pi}\left(e^{0} - e^{-j\pi(-1)}\right)\right)$$

$$= \frac{1}{4}\left(1 - \frac{1}{j\pi}(1-(-1))\right) = \frac{1}{4}\left(1 - \frac{2}{j\pi}\right)$$

$$= \frac{1}{4}\left(1 + j\frac{2}{\pi}\right) = C_1$$

For $k \geq 2$,

$$= \frac{1}{4}\int_{-1}^{0}\left(e^{j\frac{\pi}{2}(1-k)t} + e^{-j\frac{\pi}{2}(1+k)t}\right) dt$$

$$= \frac{1}{4}\left[\frac{e^{j\frac{\pi}{2}(1-k)t}}{j\frac{\pi}{2}(1-k)} + \frac{e^{-j\frac{\pi}{2}(1+k)t}}{-j\frac{\pi}{2}(1+k)}\right]_{-1}^{0}$$

$$= \frac{1}{j2\pi(1-k^2)}\left((1+k)e^{j\frac{\pi}{2}(1-k)t} - (1-k)e^{-j\frac{\pi}{2}(1+k)t}\right)_{-1}^{0}$$

$$= \frac{j}{2\pi(k^2-1)}\left[2k - k\left(e^{j\frac{\pi}{2}(k-1)} + e^{j\frac{\pi}{2}(k+1)}\right)\right]$$

$$\underbrace{e^{j\frac{\pi}{2}k}(-j) + e^{j\frac{\pi}{2}k}(+j)}_{=0}$$

$$= j\frac{k}{\pi(k^2-1)} = c_k, \quad k \geq 2$$

$k=2$, $\quad c_2 = j\frac{2}{\pi\cdot 3} = j\frac{2}{3\pi}$

$k=3$, $\quad c_3 = j\frac{3}{\pi\cdot 8} = j\frac{3}{8\pi}$

$k=4$, $\quad c_4 = j\frac{4}{\pi\cdot 15} = j\frac{4}{15\pi}$