ECE103 Lecture 13, Oct 26, 2018

Fourier Transform (FT)

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Laplace Transform $F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$

Fourier Transform $F(\omega) = F(s)\big|_{s=j\omega}$

Derivation:

periodic $f(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{jk\omega_0 t}$

where $C_k = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}\, dt$

- Quiz 4 on Monday, Oct 27 (Fourier Transform)
- HW 5 assignment on Monday, Oct 27
- Midterm Exam on Wednesday, Oct. 31

You may use ~~formulas~~ to check your mathematical derivations such as

C_k^x, C_k^y

$$y(t) = C_0^y + \sum_{k=1}^{\infty} C_k^y\, e^{jk\omega_0 t}$$

$$= C_0^y + \sum_{k=1}^{\infty} 2|C_k^y| \cos(k\omega_0 t + \theta_k^y)$$

$$C_k^y = |C_k^x|\,|H(j\omega = k\omega_0)|\, e^{j(\theta_k^x + \angle H(jk\omega_0))}$$

Note that $\omega_0 = 2\pi f_0 = 2\pi\left(\frac{1}{T_0}\right)$

In the limit as $T_0 \to \infty$

$$\lim_{T_0\to\infty} \omega_0 = \lim_{T_0\to\infty} \frac{2\pi}{T_0} = d\omega$$

$$k\omega_0 = \omega$$

For $C_k = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}$

$$\lim_{T_0\to\infty} C_k = \lim_{T_0\to\infty} \frac{1}{2\pi}\underbrace{\frac{2\pi}{T_0}}_{d\omega}\int_{-T_0/2}^{T_0/2} f(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \left[\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt}_{=F(\omega)}\right] d\omega$$

thus, $f(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{+jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} F(\omega)\, e^{+jk\omega_0 t}\, d\omega$

Since $\lim_{T_0\to\infty} \frac{2\pi}{T_0} = d\omega$

$k\omega_0 = \omega$

we can rewrite as (summation is changed in integral):

$$f(t) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{+j\omega t}\, d\omega}$$

$= FT^{-1}[FT(f(t))]$ = Inverse Fourier Transformation

where $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$ Fourier Transform

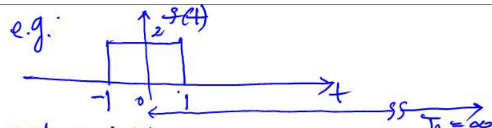
$$\boxed{f(t)} \xrightarrow{FT} \boxed{\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = F(\omega)}$$

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega = f(t)} \xleftarrow{\text{Inverse FT } (=FT^{-1})}$$

Here $f(t)$ is non-periodic (or Periodic with $T_0 = \infty$)

* later, we will handle periodic signals *

e.g.

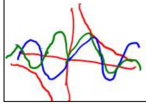


not periodic, but can also be considered periodic with period $T_0 = \infty$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-1}^{1} 2\, e^{-j\omega t}\, dt$$

$$= 2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-1}^{1} = \frac{2}{-j\omega}\left(e^{-j\omega} - e^{+j\omega}\right)$$

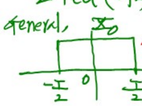
$$= 2\cdot 2\,\frac{e^{+j\omega} - e^{-j\omega}}{2j\omega} = 4\,\frac{\sin\omega}{\omega} = 4\,\text{sinc}\,\omega$$



$f(t)$ $\quad$ $F(\omega)$

$2\,\text{Rect}(t|2) \longleftrightarrow 4\,\text{sinc}(\omega)$

In General,



$f(t) = X_0\,\text{Rect}(t|T) \longrightarrow ?$ $\boxed{X_0 T\,\text{sinc}\left(\omega\frac{T}{2}\right)}$

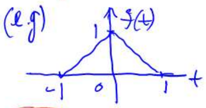
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= X_0\int_{-T/2}^{T/2} e^{-j\omega t}\, dt$$

$$= X_0\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{T/2} = X_0\left[\frac{e^{-j\omega\frac{T}{2}} - e^{+j\omega\frac{T}{2}}}{-j\omega\frac{T}{2}\,2}\right]T$$

$$= X_0 T\,\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}} = X_0 T\,\text{sinc}\left(\omega\frac{T}{2}\right)$$

(e.g)



$f(t) \xrightarrow{FT} ?$

$\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$\int e^{jk\omega t} = \frac{e^{jk\omega t}}{jk\omega}$

$= \int_{-1}^{0} (t+1) e^{-j\omega t} dt + \int_{0}^{1} (-t+1) e^{-j\omega t} dt$

$= \int_{-1}^{0} t e^{-j\omega t} dt + \int_{-1}^{0} e^{-j\omega t} dt - \int_{0}^{1} t e^{-j\omega t} dt + \int_{0}^{1} e^{-j\omega t} dt$

$= \int_{-1}^{0} t e^{-j\omega t} dt - \int_{0}^{1} t e^{-j\omega t} dt + \underbrace{\int_{-1}^{1} e^{-j\omega t} dt}_{= \frac{e^{-j\omega t}}{-j\omega}\Big|_{-1}^{1}}$

here $\int_a^b t e^{-j\omega t} dt = \int_a^b t\, d\left(\frac{e^{-j\omega t}}{-j\omega}\right)$

$= t\left(\frac{e^{-j\omega t}}{-j\omega}\right)\Big|_a^b - \int_a^b \left(\frac{e^{-j\omega t}}{-j\omega}\right) dt$

$= t\frac{e^{-j\omega t}}{-j\omega}\Big|_{-1}^{0} - \int_{-1}^{0} \frac{e^{-j\omega t}}{-j\omega} dt - t\frac{e^{-j\omega t}}{-j\omega}\Big|_{0}^{1} + \int_{0}^{1} \frac{e^{-j\omega t}}{-j\omega} dt$

$+ \frac{e^{-j\omega t}}{-j\omega}\Big|_{-1}^{1}$

$= 0 - (-1)\frac{e^{+j\omega}}{-j\omega} - \int_{-1}^{0} \frac{e^{-j\omega t}}{-j\omega} dt - 1\cdot\frac{e^{-j\omega}}{-j\omega} + 0 + \int_{0}^{1} \frac{e^{-j\omega t}}{-j\omega} dt$

$+ \frac{1}{-j\omega}\left(e^{-j\omega} - e^{+j\omega}\right)$

$= \cancel{\frac{e^{+j\omega}}{-j\omega}} - \frac{e^{-j\omega t}}{(-j\omega)^2}\Big|_{-1}^{0} - \cancel{\frac{e^{-j\omega}}{-j\omega}} + \frac{e^{-j\omega t}}{(-j\omega)^2}\Big|_{0}^{1}$

$+ \cancel{\frac{e^{-j\omega}}{-j\omega}} - \cancel{\frac{e^{+j\omega}}{-j\omega}} = \frac{e^{j\omega}-1}{(-j\omega)^2} + \frac{e^{-j\omega}-1}{(-j\omega)^2}$

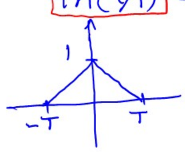
Numerator $e^{j\omega} + e^{-j\omega} - 2 = \left(e^{j\omega/2} - e^{-j\omega/2}\right)^2$ $\quad (a-b)^2 = a^2+b^2-2ab$

$= \frac{\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)^2}{(-j\omega)^2} = \frac{\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)^2}{\left(2j\cdot\frac{\omega}{2}\right)^2} = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2$

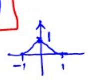
$= \left(\text{sinc}\,\frac{\omega}{2}\right)^2$

In Table 5.2, for width of 2T

$\boxed{tri(t/T)} \xrightarrow{FT} \boxed{T\,\text{sinc}^2\left(\omega\frac{T}{2}\right)}$



In the above example



$T=1$

$tri(t/1) = 1\,\text{sinc}^2\left(\frac{\omega}{2}\right)$ ✓

Sufficient Condition for Fourier Transform of $f(t)$

(Dirichlet Condition) on any time interval

i) $f(t)$ is bounded, that is $|f(t)| < M$

ii) $f(t)$ has a finite number of maxima and minima

iii) $f(t)$ has a finite number of discontinuities

iv) $f(t)$ is absolutely integrable, i.e.

$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$

"All physically realizable signals have Fourier Transforms"

e.g. $\delta(t) \xrightarrow{F} ?$ $\boxed{1}$

$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) \underbrace{e^{-j\omega(0)}}_{=1} dt$

$= \int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$F(\delta(t)) = 1$

$A\,\delta(t-t_0) \xrightarrow{F} ?$

$\int_{-\infty}^{\infty} A\delta(t-t_0) e^{-j\omega t} dt = \int_{-\infty}^{\infty} A\delta(t-t_0) e^{-j\omega t_0} dt$

$= A e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} \delta(t-t_0)\, dt}_{=1}$

$= A e^{-j\omega t_0}$

e.g. $e^{j\omega_0 t} \xrightarrow{F} ?$ $\boxed{2\pi\,\delta(\omega-\omega_0)}$

$\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt$

$= \frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)}\Big|_{-\infty}^{\infty} = ?$

On the other hand,

Inverse Fourier Transform (FT^{-1}) of $\delta(\omega-\omega_0)$ is

$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{+j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{+j\omega_0 t} d\omega$

$= \frac{1}{2\pi} e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, d\omega}_{=1} = \frac{1}{2\pi} e^{j\omega_0 t}$

thus $FT\left(e^{j\omega_0 t}\right) = 2\pi\,\delta(\omega-\omega_0)$

$$A\cos\omega_0 t \xrightarrow{FT} ?$$

$$\int_{-\infty}^{\infty} A\underbrace{\cos\omega_0 t}_{=\left(\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right)} e^{-j\omega t}\,dt \;\longrightarrow\; \frac{A}{2}\,2\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

$$= A\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

$$FT\left(A\cos\omega_0 t\right) = ?$$

$$A\cos\omega_0 t = \frac{A}{2}\left(e^{+j\omega_0 t}+e^{-j\omega_0 t}\right).$$

$$\frac{A}{2}e^{j\omega_0 t} \longrightarrow \frac{A}{2}\left[2\pi\,\delta(\omega-\omega_0)\right]$$

$$\frac{A}{2}e^{-j\omega_0 t} \longrightarrow \frac{A}{2}\left[2\pi\,\delta(\omega+\omega_0)\right]$$

thus $A\cos\omega_0 t \longrightarrow A\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$

$$B\sin\omega_0 t = \frac{B}{2j}\left(e^{+j\omega_0 t}-e^{-j\omega_0 t}\right)$$

$$\to \frac{B\pi}{j}\left(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right)$$

$$FT\left[f(-t)\right]$$

$$\int_{-\infty}^{\infty} f(-t)\,e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(-t)\,e^{-j(-\omega)(-t)}\,dt$$

$$\underset{\tau=-t}{=} -\int_{\infty}^{-\infty} f(\tau)\,e^{-j(-\omega)\tau}\,d\tau = \int_{-\infty}^{\infty} f(\tau)\,e^{-j(-\omega)\tau}\,d\tau$$

$$= F(-\omega)$$

$$f(at) \xrightarrow{FT} ?$$

$$\int_{-\infty}^{\infty} f(at)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\,e^{-j\frac{\omega}{a}\tau}\,\frac{d\tau}{a}$$

$\tau = at$

$dt = \frac{1}{a}d\tau$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} f(\tau)\,e^{-j\left(\frac{\omega}{a}\right)\tau}\,d\tau$$

$$= \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

$$g(t) = \frac{d}{dt}f(t) \xrightarrow{FT} ?$$

$$\int_{-\infty}^{\infty}\frac{d}{dt}f(t)\,e^{-j\omega t}\,dt$$

$$= f(t)\,e^{-j\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-j\omega)\,e^{-j\omega t}\,dt$$

$$= 0 + j\omega\int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt = j\omega F(\omega)$$

If $f(t)$ is Fourier transformable

$f(t) \to 0$ as $t \to \pm\infty$

$$\frac{d^n}{dt^n}f(t) \longrightarrow (j\omega)^n F(\omega)$$

$$g(t) = f_1(t) * f_2(t) \xrightarrow{FT} ? \quad \boxed{F_1(\omega)F_2(\omega)}$$

$$\int_{-\infty}^{\infty} g(t)\,e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f_1(t-\tau)f_2(\tau)\,d\tau\right)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t-\tau)f_2(\tau)\,d\tau\; e^{-j\omega(t-\tau)}e^{-j\omega\tau}\,dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t-\tau)\,e^{-j\omega(t-\tau)}\,d(t-\tau)\;\; f_2(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} f_1(t-\tau)\,e^{-j\omega(t-\tau)}\,d(t-\tau)\cdot\int_{-\infty}^{\infty} f_2(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$= F_1(\omega)\cdot F_2(\omega)$$

Convolution in time domain $\longrightarrow$ Multiplication in Freq. domain

$$F_1(\omega) * F_2(\omega) \xrightarrow{\text{Inverse FT}} ? \qquad \boxed{\frac{1}{2\pi} f_1(t) f_2(t)}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[F_1(\omega) * F_2(\omega)\right] e^{+j\omega t}\, d\omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F_1(\omega-z)\, F_2(z)\, dz\right) e^{+j\omega t}\, d\omega$$

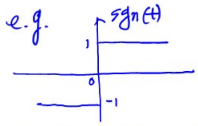
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\omega-z)\, F_2(z)\, dz\, e^{+j(\omega-z)t} e^{+jzt}\, d(\omega-z)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(z)\, e^{+j\omega z}\, dz \int_{-\infty}^{\infty} F_1(\omega-z)\, e^{+j(\omega-z)}\, d(\omega-z)$$

$$= 2\pi\ f_2(t)\cdot f_1(t) = 2\pi\ f_1(t)\, f_2(t)$$

thus $\boxed{f_1(t)\, f_2(t) \longleftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega)}$

e.g.



$$g(t) = \frac{d}{dt}\, sgn(t) = 2\,\delta(t) \xrightarrow{\text{FT}} 2$$

Also $\frac{d}{dt}\left[sgn(t) + I\right] = 2\,\delta(t)$

$$\int g(t)\, dt \xrightarrow{\text{FT}} \boxed{\frac{1}{j\omega}\, 2}$$

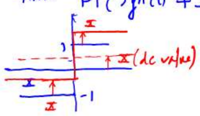
also $\boxed{\frac{2}{j\omega} + k\,\delta(\omega)}$

dc value

Here dc value of $sgn(t) = 0$,
thus $k = 0$

But when $f(t) = sgn(t) + I$

then $FT(sgn(t) + I) = \frac{2}{j\omega} + \underbrace{I\, 2\pi}_{=k}\, \delta(\omega)$



(e.g.) $u(t) = \frac{1}{2}\left[sgn(t) + 1\right]$

$$FT[u(t)] = FT\left[\tfrac{1}{2}\, sgn(t)\right] + FT\left[\tfrac{1}{2}\right]$$

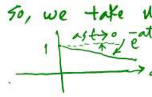
$$= \frac{1}{j\omega} + \pi\,\delta(\omega)$$

Another way of finding FT of $u(t)$ (direct method)

$$\int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt = \int_0^{\infty} 1\, e^{-j\omega t}\, dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^{\infty}$$

$$= \frac{1}{j\omega}\left(1 - e^{-j\omega(\infty)}\right) \text{ not defined !}$$

So, we take $u(t)$ as $\left(\lim_{a\to 0} e^{-at} u(t)\right)$

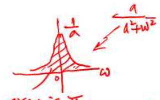


$$\int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt = \lim_{a\to 0}\int_0^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \lim_{a\to 0}\int_0^{\infty} e^{-(a+j\omega)t}\, dt = \lim_{a\to 0}\left.\frac{e^{-j(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty}$$

$$= \lim_{a\to 0}\frac{1}{a+j\omega}\left(1 - e^{-j(a+j\omega)\infty}\right) = \lim_{a\to 0}\frac{1}{a+j\omega}$$

$$= \lim_{a\to 0}\frac{a-j\omega}{(a+j\omega)(a-j\omega)} = \lim_{a\to 0}\left[\frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}\right]$$



area is π

$$\int_{-\infty}^{\infty}\frac{a}{a^2+\omega^2}\, d\omega = \left.\tan^{-1}\frac{\omega}{a}\right|_{-\infty}^{\infty} = \pi$$

$$= \lim_{a\to 0}\frac{a}{a^2+\omega^2} + \frac{1}{j\omega} = \boxed{\pi\,\delta(\omega) + \frac{1}{j\omega}}$$

$$\frac{dF(\omega)}{d\omega} \xrightarrow{\text{Inverse FT}} ? \qquad \boxed{-\frac{1}{jt} f(t)}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dF(\omega)}{d\omega}\, e^{+j\omega t}\, dt$$

$$= \frac{1}{2\pi}\left[F(\omega)\, e^{j\omega t} - \int_{-\infty}^{\infty} F(\omega)\, \frac{e^{j\omega t}}{jt}\, d\omega\right]_{-\infty}^{\infty}$$

$$= \left(-\frac{1}{jt}\right) f(t)$$