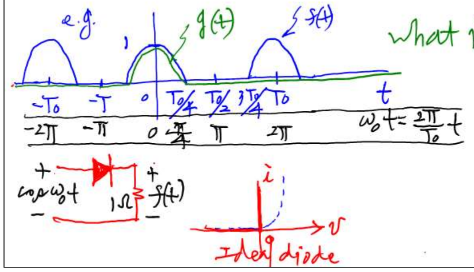
ECE 103 Fall 2018 Lecture 14, Oct 29, 2018

Generating function of $f(t)$, called $g(t)$, is

$$g(t) = \begin{cases} f(t), & -T < t < T, \quad T = \frac{T_0}{2} \\ 0, & \text{elsewhere} \end{cases}$$

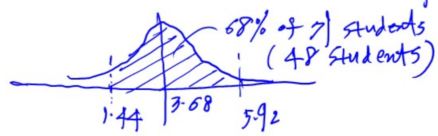


What is $G(\omega)$?

Quiz 3

Average of 71 papers = 3.68

σ = 2.24



Group tutoring on Oct 30 (T) 3-7pm by Dan Li in McHenry 3374.

$$f(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

and $\sum_{k=-\infty}^{\infty} \delta(t-kT_0) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$ Fourier series

Fourier Transform $\mathcal{F}\left(\sum_{k=-\infty}^{\infty} \delta(t-kT_0)\right) = \mathcal{F}\left(\sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t}\right)$

(since $C_k = \frac{1}{T_0}\int_{T_0} \delta(t-kT_0)\, dt = \frac{1}{T_0}$)

$$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} \mathcal{F}\left(e^{+jk\omega_0 t}\right)$$

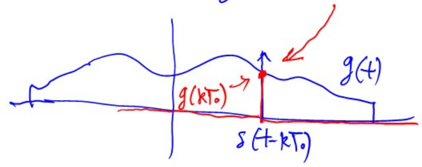
$$= \frac{1}{T_0}\sum_{k=-\infty}^{\infty} \boxed{2\pi\,\delta(\omega - k\omega_0)}$$

thus $F(\omega) = G(\omega) \cdot \frac{1}{T_0}\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - k\omega_0)$

$$= \boxed{\frac{2\pi}{T_0} G(\omega) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \omega_0 \sum_{k=-\infty}^{\infty} G(k\omega_0)\,\delta(\omega - k\omega_0)}$$

ω_0

$$\delta(t-kT_0)\, g(t) = \delta(t-kT_0)\, g(kT_0)$$



$$\mathcal{F}\sum_{k=-\infty}^{\infty} \delta(t-kT_0) = \sum_{k=-\infty}^{\infty} \mathcal{F}\left[\delta(t-kT_0)\right]$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} \delta(t-kT_0)\, e^{-j\omega t}\, dt}$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(k-T_0)\, e^{-jk\omega T_0}\, dt = \sum_{k=-\infty}^{\infty} e^{-j\omega k T_0} = \sum_{k=-\infty}^{\infty} e^{jk\omega T_0}$$

Another form

then, $C_k e^{jk\omega_0 t} \overset{?}{=} \omega_0 \underbrace{\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)}$

Fourier series with period ω_0

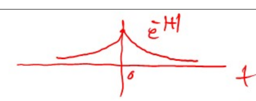
$\Downarrow$

$$C_k = \frac{1}{\omega_0}\int_{-\omega_0/2}^{\omega_0/2} \delta(\omega)\, d\omega = \frac{1}{\omega_0} \text{ for all } k.$$

thus, $\omega_0 \sum_{k=-\infty}^{\infty} \frac{1}{\omega_0} e^{jk(T_0)\omega} = \sum_{k=-\infty}^{\infty} e^{jkT_0\omega}$

Yes same

(e.g.)



Fourier transform of $x(t) = e^{-|t|}$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{+t} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt$$

$$= \left.\frac{e^{(1-j\omega)t}}{1-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{1-j\omega} - 0 + 0 + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2}$$

and

$$\mathcal{F}\left[\frac{d}{dt} e^{-|t|}\right] = (j\omega)\, \mathcal{F}\left[e^{-|t|}\right] = j\omega \frac{2}{1+\omega^2}$$

Table 5-1

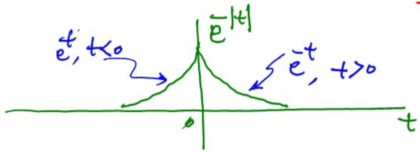
$\frac{d^n}{dt^n} f(t) \leftrightarrow (j\omega)^n F(\omega)$

Alternate way: $\int_{-\infty}^{\infty}\left(\frac{d}{dt}e^{-|t|}\right)e^{-j\omega t}dt = \int_{-\infty}^{0}\left(\frac{d}{dt}e^{t}\right)e^{-j\omega t}dt + \int_{0}^{\infty}\left(\frac{d}{dt}e^{-t}\right)e^{-j\omega t}dt = \int_{-\infty}^{0}e^{t}e^{-j\omega t}dt - \int_{0}^{\infty}e^{-t}e^{-j\omega t}dt$

$$= \frac{e^{(1-j\omega)t}}{(1-j\omega)}\Big|_{-\infty}^{0} - \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\Big|_{0}^{\infty}$$

$$= \left(\frac{1}{1-j\omega} - 0\right) - \left(0 - \frac{1}{-(1+j\omega)}\right)$$

$$= \frac{1}{1-j\omega} - \frac{1}{1+j\omega} = \frac{1+j\omega-1+j\omega}{1+\omega^2} = j\omega\frac{2}{1+\omega^2}$$



$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t} \quad \boxed{\text{(periodic case)}}$$

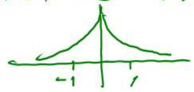
$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} c_k (jk\omega_0) e^{+jk\omega_0 t}$$

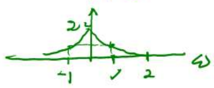
As $T_0 \to \infty$ ($\omega_0 \to 0$)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega t}, \quad \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} c_k (j\omega) e^{+j\omega t} = (j\omega)\sum_{k=-\infty}^{\infty} c_k e^{+j\omega t}$$

$$x(t) \xrightarrow{\mathcal{F}} X(\omega) \qquad \frac{dx(t)}{dt} \xrightarrow{\mathcal{F}} j\omega X(\omega)$$

For $x(t) = e^{-|t|} \longleftrightarrow X(\omega) = \frac{2}{1+\omega^2}$





$$\hat{x}(t) = e^{-|t|} \cdot \cos\omega_0 t \xrightarrow{\mathcal{F}} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] * \frac{2}{1+\omega^2}$$

$$\cos\omega_0 t = \left[\frac{e^{+j\omega_0 t}+e^{-j\omega_0 t}}{2}\right] \xrightarrow{\mathcal{F}} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

$$= 2\pi\left[\frac{1}{1+(\omega-\omega_0)^2} + \frac{1}{1+(\omega+\omega_0)^2}\right]$$

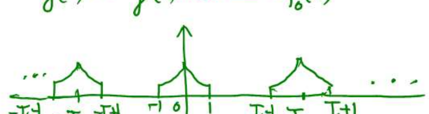
$$\int_{-\infty}^{\infty}\delta(z-\omega_0)\frac{2}{1+(\omega-z)^2}dz = \int_{-\infty}^{\infty}\delta(z-\omega_0)\frac{2}{1+(\omega-\omega_0)^2}dz$$

$$\mathcal{F}\left[e^{j\omega_0 t}\right] = \int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty}e^{-j(\omega-\omega_0)t}dt = \frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)}\Big|_{-\infty}^{\infty}$$

$$e^{-j(\omega-\omega_0)\infty} = ?$$

$$2\pi \times 1 \longleftarrow \frac{1}{2\pi}\int \underbrace{F(\omega)}_{\delta(\omega-\omega_0)} e^{+j\omega t}d\omega$$

$$y(t) = g(t) * \sum \delta_{T_0}(t)$$



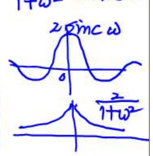
$$g(t) = e^{-|t|}\,\text{rect}(t/T)$$

$$Y(\omega) = G(\omega) \cdot \Delta_{T_0}(\omega)$$

$$\frac{1}{T_0}\sum 2\pi\delta(\omega-k\omega_0) = \omega_0\sum\delta(\omega-k\omega_0)$$

$$= \omega_0\sum G(k\omega_0)\,\delta(\omega-k\omega_0)$$

$$G(\omega) = \mathcal{F}\left[e^{-|t|}\right] * \mathcal{F}\left[\text{rect}(t/2)\right] = \frac{2}{1+\omega^2} * 2\,\text{sinc}\,\omega$$

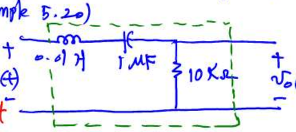


Bode plot of $H(\omega)$

section 5.4 pp. 245 - 248 (Textbook)

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega) \qquad H(\omega) = |H(\omega)|\,\angle H(\omega) = |H(\omega)|\,e^{j\angle H(\omega)}$$

(Example 5.20)



$$H(\omega) = \frac{10^4}{j0.01\omega + \frac{1}{j\omega(10^{-6})} + 10^4} = \frac{j\omega 10^{-2}}{1-10^{-8}\omega^2 + j\omega 10^{-2}}$$

$$|H(\omega)| = \frac{10^{-2}\omega}{\sqrt{(1-10^{-8}\omega^2)^2+(10^{-2}\omega)^2}} \qquad \angle H(\omega) = 90^\circ - \tan^{-1}\frac{10^{-2}\omega}{1-10^{-8}\omega^2}$$

At $\omega=0$ $\quad H(\omega=0) = \dfrac{10^4}{j0.01(0) + \frac{1}{j(0)10^{-6}} + 10^4} = 0$

$\omega = 10^4$, $\quad |H(\omega)| = \dfrac{10^{-2}(10^4)}{\sqrt{(1-10^{-8}(10^4)^2)^2 + (10^{-2}10^4)^2}}$

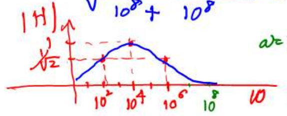
$= \dfrac{10^{-2}(10^4)}{\sqrt{0 + 10^4}} = 1$

$\omega = 10^6$, $\quad H(\omega) = \dfrac{10^{-2}(10^6)}{\sqrt{(1-10^{-8}(10^6)^2)^2 + (10^{-2}10^6)^2}}$

$= \dfrac{10^4}{\sqrt{10^8 + 10^8}} = \dfrac{1}{\sqrt{2}}$

$\omega = 10^2$

$\dfrac{10^{-2}(10^2)}{\sqrt{[1-10^{-8}(10^2)^2]^2 + [10^{-2}(10^2)]^2}} = \dfrac{1}{\sqrt{2}}$



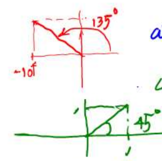
$\omega = 10^8$ $\quad \dfrac{10^{-2}(10^8)}{\sqrt{[(1-10^{-8}(10^8)^2]^2 + [10^{-2}(10^8)]^2}} = 10^{-2}$

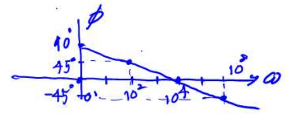
$\angle H(\omega)$, $\quad$ At $\omega = 0$ $\quad \angle = 90^\circ - \tan^{-1}\dfrac{10^{-2}\omega}{1-10^{-8}\omega^2} = 90^\circ$

$\omega = 10^4$ $\quad 90^\circ - \tan^{-1}\dfrac{10^2}{1-10^{-8}(10^4)^2} = 0$

$\omega = 10^6$ $\quad 90^\circ - \tan^{-1}\dfrac{10^{-2}(10^6)}{1-10^{-8}(10^6)^2} = -45^\circ$

$\omega = 10^2$ $\quad 90^\circ - \tan^{-1}\dfrac{10^{-2}(10^2)}{1-10^{-8}(10^2)^2} = 45^\circ$





$$\boxed{\begin{array}{l}\text{Parseval's Theorem} \\ \displaystyle\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega\end{array}}$$

$f(t) = \frac{1}{2\pi}\int F(\omega)e^{+j\omega t}\,dt$

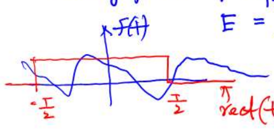
Proof $\quad \displaystyle\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} f(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{+j\omega t}\,d\omega\right)dt$

$\displaystyle = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\underbrace{\int_{-\infty}^{\infty} f(t)e^{+j\omega t}\,dt}_{F(-\omega)=F^*(\omega)}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$ ⊠

$\Rightarrow |F(\omega)| = |F(-\omega)|$

Average Power $\quad \displaystyle P = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}|f_T(t)|^2\,dt$

$\displaystyle E = \int_{-\infty}^{\infty}|f_T(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$



$\displaystyle P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|f_T(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{\infty}|F_T(\omega)|^2\,d\omega$

$\displaystyle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\lim_{T\to\infty}\frac{1}{T}|F_T(\omega)|^2}_{P_f(\omega)}\,d\omega$

$\displaystyle P_f(\omega) = \lim_{T\to\infty}\frac{1}{T}|F_T(\omega)|^2$ $\quad$ (Power spectral density)

Power $\displaystyle P = \frac{1}{2\pi}\int_{-\infty}^{\infty}P_f(\omega)\,d\omega$

$\displaystyle = \sum_{k=-\infty}^{\infty}|c_k|^2 = \frac{1}{4\pi^2}F(0) + \frac{\overset{1}{\cancel{1\times 2}}}{\underset{2}{\cancel{4}}\pi^2}\sum_{k=1}^{\infty}|F(k\omega_0)|^2$

$\displaystyle \sum c_k e^{+jk\omega_0 t} \longleftrightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}F(k\omega_0)\delta(\omega - k\omega_0)$

$c_k = \frac{1}{2\pi}F(k\omega_0)$

$|F(k\omega_0)| = |F(-k\omega_0)|$ complex conjugates have same magnitude.

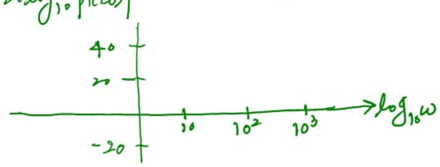
$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$

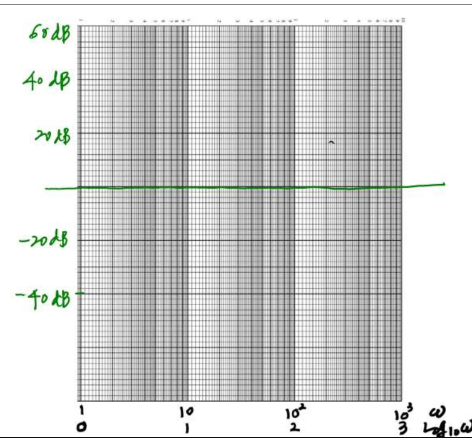
$Y(\omega) = H(\omega)X(\omega)$

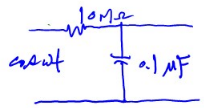
$|Y(\omega)|^2 = |H(\omega)|^2|X(\omega)|^2$

$10\log_{10}|Y(\omega)|^2 = \underbrace{20\log_{10}|H(\omega)|}_{[dB]} + 20\log_{10}|X(\omega)|$

$20\log_{10}|H(\omega)|$







$$H(\omega) = \frac{\frac{R}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC}$$

For $R = 1$, $H(\omega) = \frac{1}{1 + j\omega(10^{7} \cdot 10^{-7})} = \frac{1}{1 + j\omega}$

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

$\omega = 1$ $\quad \frac{1}{\sqrt{2}} \quad 20\log 2^{-\frac{1}{2}} = -10\log_{10} 2 = -3\text{ dB}$

$10 \quad \frac{1}{10} \quad 20\log_{10} 10^{-1} = -20\text{ dB}$

$10^2 \quad \frac{1}{10^2} \quad 20\log_{10} 10^{-2} = -40\text{ dB}$

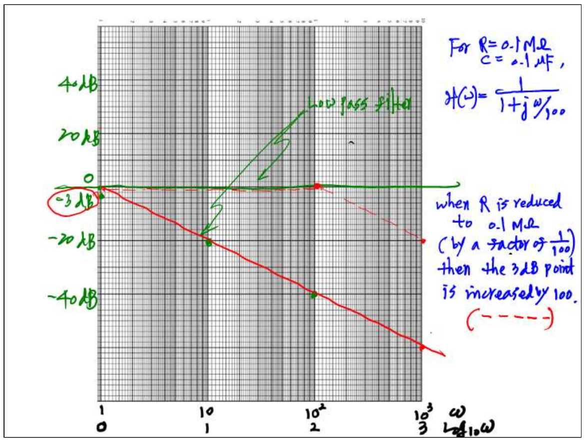


Table 5.1 Fourier Transform Properties

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time reversal	$f(-t)$	$F(-\omega)$
Time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$f(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega - \omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
Modulation (Multiplication)	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega) * F_2(\omega)$
Integration	$\int_{-\infty}^{t} f(\tau)\,d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\,\delta(\omega)$
Differentiation in time	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
Differentiation in Frequency	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Symmetry	$f(t)$ real	$F(-\omega) = F^*(\omega)$

Table 5.2 Fourier Transform Pairs

Time Domain Signal	Fourier Transform
$f(t)$	$\int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}dt$
$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}d\omega$	$F(\omega)$
$\delta(t)$	1
$A\delta(t - t_0)$	$Ae^{-j\omega t_0}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
K	$2\pi K\delta(\omega)$
$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0)+(\omega + \omega_0)]$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$\mathrm{rect}(t/T)$	$T\mathrm{sinc}(\omega T/2)$
$\cos(\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$

Inverse FT

$\mathrm{rect}(t/T)\cos(\omega_0 t)$	$\frac{T}{2}\left[\mathrm{sinc}\left(\frac{(\omega-\omega_0)T}{2}\right) + \mathrm{sinc}\left(\frac{(\omega+\omega_0)T}{2}\right)\right]$
$\frac{\beta}{\pi}\mathrm{sinc}(\beta t)$	$\mathrm{rect}(\omega/2\beta)$
$\mathrm{tri}(t/T)$	$T\,\mathrm{sinc}^2(T\omega/2)$
$\mathrm{sinc}^2(Tt/2)$	$\frac{2\pi}{T}\mathrm{tri}(\omega/T)$
$e^{-at}u(t)$, $\mathrm{Re}\{a\} > 0$	$\frac{1}{a+j\omega}$
$te^{-at}u(t)$, $\mathrm{Re}\{a\} > 0$	$\left(\frac{1}{a+j\omega}\right)^2$
$t^{n-1}e^{-at}u(t)$, $\mathrm{Re}\{a\} > 0$	$\frac{(n-1)!}{(a+j\omega)^n}$
$e^{-a\lvert t\rvert}$, $\mathrm{Re}\{a\} > 0$	$\frac{2a}{a^2+\omega^2}$
$\sum_{n=-\infty}^{\infty} g(t - nT_0)$	$\sum_{n=-\infty}^{\infty} \omega_0 G(n\omega_0)\,\delta(\omega - n\omega_0),\ \omega_0 = \frac{2\pi}{T_0}$
$\sum_{n=-\infty}^{\infty} g(t - nT_0) = \sum_{n=-\infty}^{\infty} C_K e^{jk\omega_0 t}$	$2\pi\sum_{n=-\infty}^{\infty} C_n\delta(\omega - n\omega_0),\ \omega_0 = \frac{2\pi}{T_0}$
$\delta_T(t)$	$\sum^{\infty} \omega_0\delta(\omega - K\omega_0)$