ECE 103 Lecture 18, Nov. 9, 2018

Quiz 6 on Nov. 14 (W)
coverage Bode plot
Derivation of $H(\omega)$

Born	April 30, 1916 Petoskey, Michigan, United States
Died	February 24, 2001 (aged 84) Medford, Massachusetts, United States
Nationality	American
Alma mater	University of Michigan, MIT
Known for	[show]
Awards	Stuart Ballantine Medal (1955) IEEE Medal of Honor (1966) National Medal of Science (1966) Harvey Prize (1972) Claude E. Shannon Award (1972) Harold Pender Award (1978) John Fritz Medal (1983) Kyoto Prize (1985) National Inventors Hall of Fame (2004)

Scientific career

Fields	Mathematics and electronic engineering
Institutions	Bell Labs

Claude Shannon (Apr. 30, 1916 – Feb 24, 2001)

Shannon was born in 1916 in Petoskey, Michigan, the son of a judge and a teacher. Among other inventive endeavors, as a youth he built a telegraph from his house to a friend's out of fencing wire. He graduated from the University of Michigan with degrees in electrical engineering and mathematics in 1936 and went to M.I.T., where he worked under computer pioneer Vannevar Bush on an analog computer called the differential analyzer.

Shannon's M.I.T. master's thesis in electrical engineering has been called the most important of the 20th century: in it the 22-year-old Shannon showed how the logical algebra of 19th-century mathematician George Boole could be implemented using electronic circuits of relays and switches. This most fundamental feature of digital computers' design--the representation of "true" and "false" and "0" and "1" as open or closed switches, and the use of electronic logic gates to make decisions and to carry out arithmetic--can be traced back to the insights in Shannon's thesis.

Nyquist-Shannon Sampling Theorem

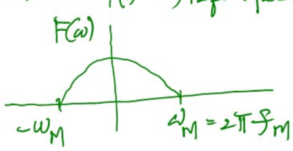
Sampling is a process of converting a signal (for example, a function of continuous time and/or space) into a numeric sequence (a function of discrete time and/or space). Shannon's version of the theorem states:[2]

> If a function x(t) contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $1/(2B)$ seconds apart.

A sufficient sample-rate is therefore anything larger than $2B$ samples per second. Equivalently, for a given sample rate f_s, perfect reconstruction is guaranteed possible for a bandlimit $B < f_s/2$.

Shannon's Sampling Theorem:

Let $f(t)$ have its max. freq. of f_M.
or its freq. spectrum is



Example

$$f(t) = 3\cos t + 5\cos 3t$$

has its $\omega_M = 3$ $\quad 2\pi f_M = 3 \quad f_M = \frac{3}{2\pi}$

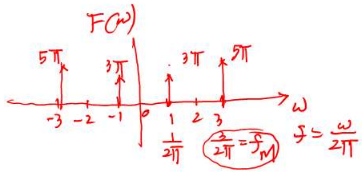
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \longrightarrow \quad 2f_M = \frac{2\times 3}{2\pi} \text{ samples/sec.}$$

$$f(t) = \boxed{3\left(\frac{e^{jt} + e^{-jt}}{2}\right)} + 5\left(\frac{e^{j3t} + e^{-j3t}}{2}\right)$$

$$FT^{-1}\big(2\pi\,\delta(\omega-1)\big) \longrightarrow \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(\omega-1)\, e^{+j\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} \delta(\omega-1)\, e^{+jt}\, d\omega = \underline{e^{jt}\cdot 1}$$

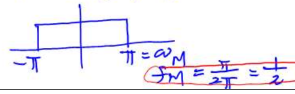
thus $F(\omega) = \frac{3}{2} 2\pi\left[\delta(\omega-1) + \delta(\omega+1)\right] + \frac{5}{2}\pi\left[\delta(\omega-3) + \delta(\omega+3)\right]$



$$\frac{\beta}{\pi} \operatorname{sinc}(\beta t) \longleftrightarrow \operatorname{rect}(\omega | 2\beta)$$

If $\beta = \pi$

$$\operatorname{sinc} \pi t \longleftrightarrow \operatorname{rect}(\omega | 2\pi)$$



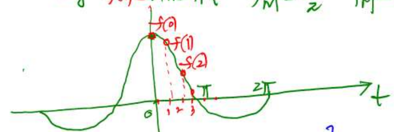
$$f_M = \frac{\pi}{2\pi} = \frac{1}{2}$$

Given f_M (or ω_M) of $f(t)$

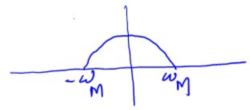
$$T_M = \frac{1}{f_M} = \frac{2\pi}{\omega_M},$$

then $f(t)$ can be sampled at every $\frac{T_M}{2}$ seconds so that $f(t)$ can be reconstructed from the set of sampled data

e.g. $f(t) = \operatorname{sinc} \pi t$ $\quad f_M = \frac{1}{2}$ $\quad T_M = 2$, sampling space $\frac{T_M}{2} = 1$



From $\{f(0), f(1), f(2), \cdots\}$ we can reconstruct $f(t)$!



$$\omega_M T_M = 2\pi$$

$$T_M = \frac{2\pi}{\omega_M} = \frac{2\pi}{2\pi f_M} = \frac{1}{f_M}$$

Suppose $\omega_M = 2\pi \times 50$

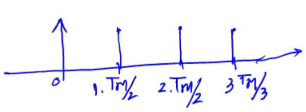
$$f_M = 50$$

$$T_M = \frac{1}{f_M} = \frac{1}{50} = 0.02 \text{ second}$$

$$\boxed{\frac{T_M}{2} = 0.01 \text{ second}}$$ or $2 \times f_M = 2 \times 50 = 100$ points/sec.

$$\frac{T_M}{2} = \frac{1}{2 f_M} \rightarrow \boxed{2 \times f_M \text{ sample points/sec}}$$

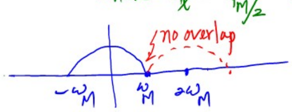
$\delta_{T_M/2}(t)$



$$f(t) \cdot \delta_{T_M/2}(t) \longleftrightarrow F(\omega) * \Delta(\omega)$$

$$\delta_{T_M/2}(t) \longleftrightarrow \sum_{-\infty}^{\infty} \omega_x \, \delta(\omega - n\omega_x)$$

where $\omega_x = \frac{2\pi}{T_M/2} = 2\omega_M$

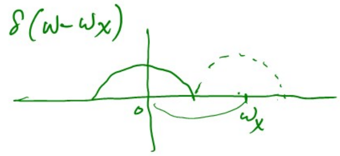


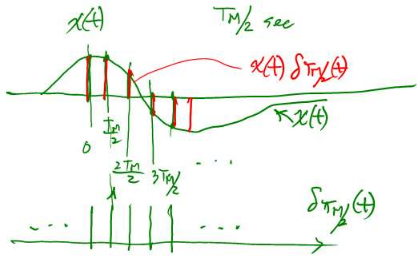
$$f(t)\, \delta(t - x)$$

$$= f(x)\, \delta(t - x)$$

$$F(\omega)\ \delta(\omega - \omega_x)$$

$$= F(\omega_x)\ \delta(\omega - \omega_x)$$

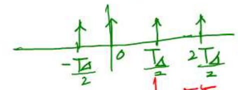




What if sampled every Δ seconds $> \frac{T_S}{2}$?

(i.e. not sufficient number of equally spaced sample points)

Instead of T_S, let us consider $T_\Delta > T_S$

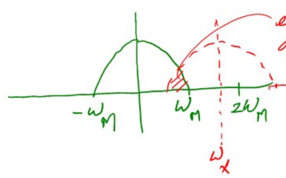


$= \delta_{T_\Delta/2}(t)$

$\downarrow$ FT

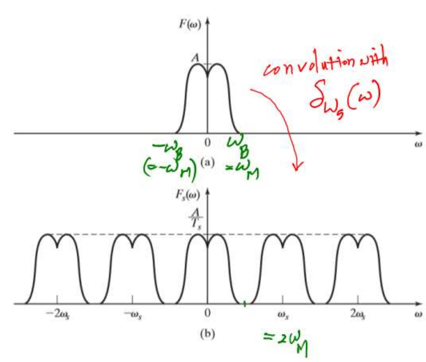
$\sum \omega_\Delta \, \delta(\omega - n\omega_\Delta)$

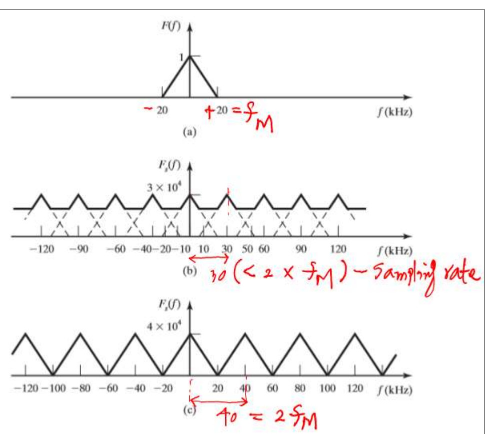
$$\boxed{\omega_\Delta = \frac{2\pi}{T_\Delta/2} < \frac{2\pi}{T_M/2} = 2\omega_M}$$



error due to overlap due to the fact that $\omega_x < 2\omega_M$

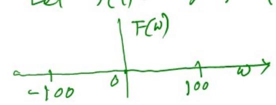
If $\omega_x > 2\omega_M$, then no error, but freq. band is not fully used.





Example

Let $f(t) = \cos 100t$ $\quad \pi = 100$



$T_M = \frac{2\pi}{\omega_M} = \frac{2\pi}{100} = 0.02\pi$

$\frac{T_M}{2} = 0.01\pi$ sec. $= 0.0314$ sec

What is sampled every 0.04 sec ?

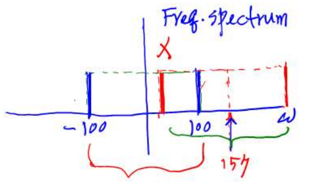
(> 0.0314 sec)

$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\,0.04)$

$\updownarrow$

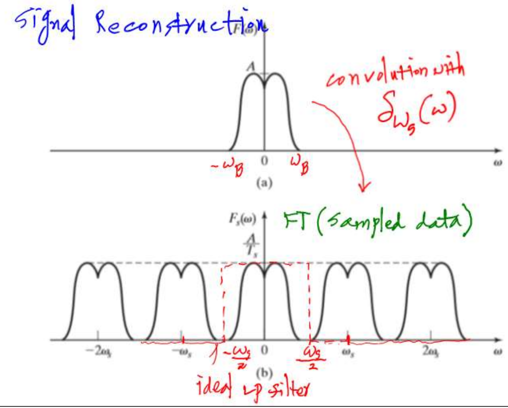
$\omega_s \, \delta(\omega - \omega_s)$

$$\omega_s = \frac{2\pi}{0.04} = 50\pi\,(=3.14) = 157$$



within the original freq. range, a new signal component is added at $157-50=107\,(=\omega)$

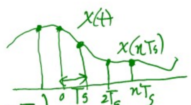
Signal Reconstruction



$$H(\omega) = T_s\ \mathrm{rect}(\omega | \omega_s)$$

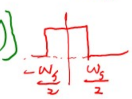
$$h(t) = \mathrm{sinc}\left(\frac{\omega_s}{2} t\right)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$$

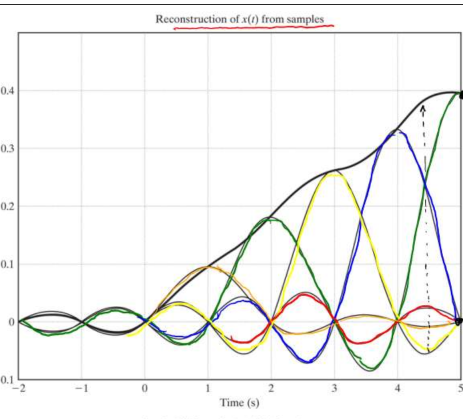


$$\hat{x}(t) = x_s(t) * h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}\left[\frac{\omega_s}{2}(t-nT_s)\right]$$



ideally should be $= x(t)$



$$f(t) = A \sin 5t + B \cos 10t + C \sin(100t + 30^\circ)$$

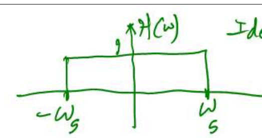
$$\omega_M = ? \quad = 100$$

$$f_M = \frac{\omega_M}{2\pi} = \frac{100}{2\pi}$$

$$T_M = \frac{1}{f_M} = \frac{2\pi}{100} = \frac{6.28}{100} = 0.0628\ \mathrm{sec}$$

$$T_{sample} = \frac{T_M}{2} = \frac{0.0628}{2} = 0.0314\ \mathrm{sec}$$

Ideal filter



$$h(t) = ?$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{+j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_s}^{+\omega_s} 1 \cdot e^{+j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{+j\omega t}}{jt}\right]_{-\omega_s}^{+\omega_s}$$

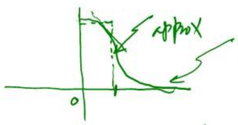
$$\sin \omega_s t$$

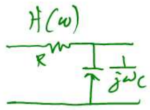
$$= \frac{1}{\cancel{2}\pi}\,\frac{1}{jt}\left[\underbrace{e^{+j\omega_s t} - e^{-j\omega_s t}}_{2}\right]\cancel{2} = \frac{1}{\pi}\left(\frac{e^{+j\omega_s t} - e^{-j\omega_s t}}{2j}\right)\frac{\omega_s}{\omega_s t}$$

$$= \frac{\omega_s}{\pi}\left(\frac{\sin \omega_s t}{\omega_s t}\right) = \boxed{\left(\frac{\omega_s}{\pi}\right) \text{sinc } \omega_s t}$$

$$\left(\begin{array}{l} \omega_s T_s = 2\pi \\ \omega_s = \frac{2\pi}{T_s} \end{array}\right) \Rightarrow = \frac{2\cancel{\pi}}{T_s \cancel{\pi}} \text{sinc } \omega_s t$$

$$= \frac{2}{T_s} \text{sinc } \omega_s t$$





$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + jRC\omega}$$

$h(t) = ?$

$$\Leftrightarrow |H(\omega)| = \frac{1}{\sqrt{1+(RC\omega)^2}}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+(RC\omega)^2}} e^{+j\omega t} d\omega = ?$$

$H(\omega) \to h(t)$

$$= \frac{1}{2\pi}\int \underbrace{\left(\frac{1}{1+j\omega RC}\right)}_{=} e^{+j\omega t} dt$$

$$\frac{1}{2\pi}\int \frac{1}{\sqrt{1+(\omega RC)^2}} e^{+j\omega t} e^{+j(-\tan \omega RC)} d\omega$$