ECE 103 Lecture 19 Nov. 14, 2018

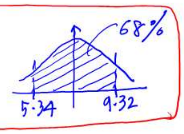
Laplace Transform (chap 7, Textbook)

~~semester~~ Also, you are welcome to download a pdf copy of Signals and Systems: Theory and Applications at:

https://www.publishing.umich.edu/publications/ee/

chapter 3

Quiz 5 Avg = 7.33
$\sigma$ = 1.99




Today's DRC Quiz (# 6) will be done in this room. not in E2-234.

# SIGNALS AND SYSTEMS:

## Theory and Applications


Fawwaz T. Ulaby
*The University of Michigan*

Andrew E. Yagle
*The University of Michigan*

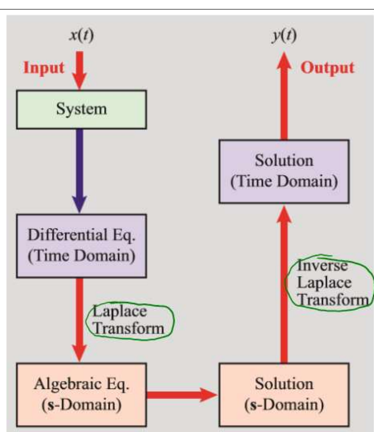

On Tue, Nov 13, 2018 at 6:30 AM Fawwaz Ulaby <ulaby@eecs.umich.edu> wrote:

Dear Steve

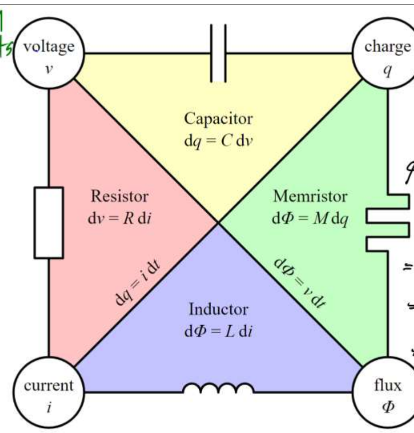
The circuits and signals and systems books are used at over 100 US universities. By converting them from the traditional for-purchase textbooks to free pdf downloads, the authors gave up about $100k in royalties, but students end up saving about $2.5 million.
The pdf is available to anyone, and most universities have bought a few hardcopies for their instructors. The hardcopy cost is for printing. Operational costs are subsidized by the university of Michigan.

Fawwaz

From Ulaby Book




RLCM elements




$\phi = f(q)$

$\frac{d\phi}{dt} = v$

$= \frac{d}{dt} f(q)$

$= \frac{df(q)}{dq} \frac{dq}{dt}$

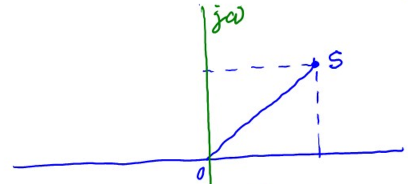
$= \boxed{M(q)}\, i$

Memristance

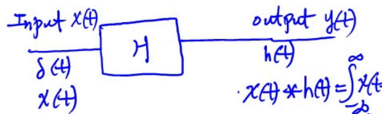
## Mathematical Models of physical Elements

| | Time Domain $t$ | Freq Domain $\omega$ (Fourier Transform) | Complex $s$ domain (Laplace Transform) |
|---|---|---|---|
| R ($i$, $+\ v\ -$) | $v = Ri$ Ohm's Law | $V(\omega) = R\,I(\omega)$ | $V(s) = R\,I(s)$ |
| C ($i$, $+\ v\ -$) | $i = C\frac{dv}{dt}$ <br> $v = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$ | $I(\omega) = j\omega C\,V(\omega)$ <br> $V(\omega) = \frac{1}{j\omega C} I(\omega)$ | $I(s) = sC\,V(s)$ <br> $V(s) = \frac{1}{sC} I(s)$ |
| L ($i$, $+\ v\ -$) | $v = L\frac{di}{dt}$ <br> $i = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau$ | $V(\omega) = j\omega L\,I(\omega)$ <br> $I(\omega) = \frac{1}{j\omega L} V(\omega)$ | $V(s) = sL\,I(s)$ <br> $I(s) = \frac{1}{sL} V(s)$ |
| | ordinary diff. eq | Algebra | Algebra |

## s-plane ($s = \sigma + j\omega$)




$$s\Big|_{\sigma=0} = j\omega \text{ (Fourier Transf.)}$$



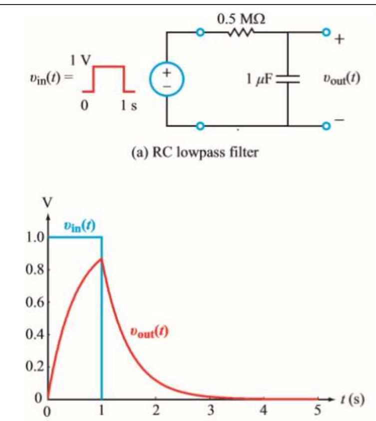

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$e^{j\omega_0 t} \longrightarrow H(\omega_0)\,e^{j\omega_0 t}$$

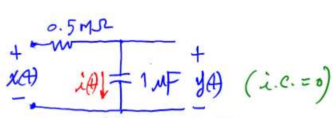
$$e^{st} \longrightarrow H(s)\,e^{st}$$

For $x(t) = e^{st}$, $y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} h(\tau)\,e^{s(t-\tau)}\,d\tau$

$$= \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau}_{} \cdot e^{st} = H(s)\cdot e^{st}$$




(a) RC lowpass filter




(i.c. = 0)

$$i(t) = C\frac{dy(t)}{dt} = 10^{-6}\frac{dy(t)}{dt}$$

$$x(t) = 0.5\times 10^{6}\left(10^{-6}\frac{dy(t)}{dt}\right) + y(t)$$

$$= 0.5\frac{dy(t)}{dt} + y(t)$$

For $x(t) = u(t)$

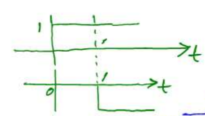
$\downarrow \mathcal{L}$

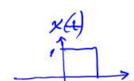
$$\frac{1}{s} = 0.5\,s\,Y(s) + Y(s)$$

$$Y(s) = \frac{1}{s(0.5s+1)} = \frac{2}{s(s+2)} = \frac{1}{s} + \frac{-1}{s+2}$$

$\uparrow \mathcal{L}^{-1}$

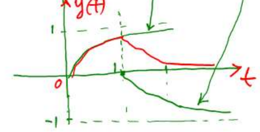
$$y(t) = u(t)\left[1 - e^{-2t}\right]$$




$$x(t) = u(t) - u(t-1)$$

$$y(t) = u(t)\left(1 - e^{-2t}\right) - u(t-1)\left(1 - e^{-2(t-1)}\right)$$



$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

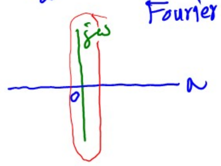
Generally $F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$

Bilateral Laplace Transform

when $s = j\omega$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Fourier Transform




For $\sigma > 0$

$$\mathcal{F}\left[f(t)\, e^{-\sigma t}\right] = \int_{-\infty}^{\infty} f(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

Fourier Transform

$$= \int_{-\infty}^{\infty} f(t)\, e^{-(\sigma + j\omega)t}\, dt = F(\sigma + j\omega)$$

$$\overset{s=\sigma+j\omega}{=} \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt = F(s)$$

Also $f(t)\, e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma + j\omega)\, e^{j\omega t}\, d\omega$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega\right] e^{-\sigma t}$$

$$f(t) \underset{s=\sigma+j\omega}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{st}\, \frac{ds}{j} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$

$ds = j\, d\omega \rightarrow d\omega = \frac{ds}{j}$

Single-sided (Unilateral) Laplace Transformation

$$\mathcal{L}[f(t)] = \int_{0}^{\infty} f(t)\, e^{-st}\, dt$$

$$\left[\ \mathcal{L}_b[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt\ \right]$$

bilateral Laplace transform

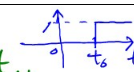
Example

$$f(t) = u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \\ \text{at } t = 0 & \text{discontinuity} \end{cases}$$

$$\mathcal{L}[u(t)] = \int_{0}^{\infty} 1\, e^{-st}\, dt = \frac{1}{-s}\, e^{-st}\Big|_{0}^{\infty}$$

$$= \frac{1}{s}\left(1 - \lim_{t\to\infty} e^{-st}\right) = \boxed{\frac{1}{s}},\ \text{for } \mathrm{Re}(s) > 0$$

Example $u(t - t_0)$




$$\mathcal{L}[u(t - t_0)] = \int_{0}^{\infty} u(t - t_0)\, e^{-st}\, dt$$

$$= \int_{t_0}^{\infty} e^{-st}\, dt = \frac{e^{-st}}{-s}\Big|_{t_0}^{\infty}$$

$$= \frac{1}{s}\left(e^{-st_0} - \lim_{t\to\infty} e^{-st}\right)$$

$$= \frac{1}{s}\, e^{-st_0} \quad \text{for } \mathrm{Re}\, s > 0$$

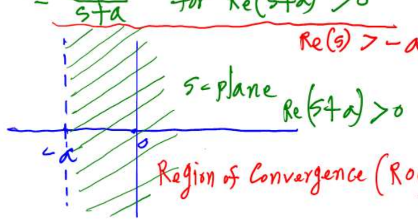
Example $f(t) = e^{-at}\, u(t)$

$$\mathcal{L}[f(t)] = F(s) = \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

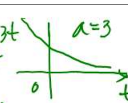
$$= \frac{e^{-(s+a)t}}{-(s+a)}\Big|_{0}^{\infty} = \frac{1}{s+a}\left(1 - \lim_{t\to\infty} e^{-(s+a)t}\right)$$

$$= \frac{1}{s+a} \quad \text{for } \mathrm{Re}(s+a) > 0$$

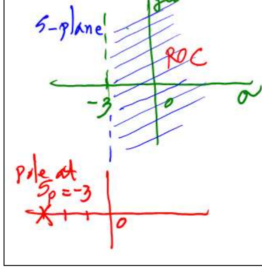
$\mathrm{Re}(s) > -a$



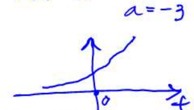

$f(t) = e^{-at} = e^{-3t}$ $\quad a = 3$



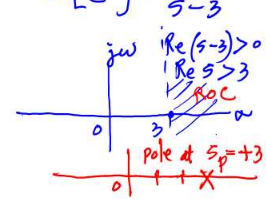

$$\mathcal{L}\left[e^{-3t}\right] = \frac{1}{s+3}$$




$f(t) = e^{-at} = e^{+3t}$ $\quad a = -3$




$$\mathcal{L}\left[e^{3t}\right] = \frac{1}{s-3}$$



$$\mathcal{L}[\cos\omega_0 t] = ?$$

$$e^{-at}\cos\omega_0 t \longrightarrow \frac{s+a}{(s+a)^2+\omega_0^2}$$

For $a=0$ $\qquad \frac{s}{s^2+\omega_0^2}$

What about direct transformation ?

Example

$$\mathcal{L}[e^{-at}\cos\omega_0 t] = ?$$

$$e^{-at}\cos\omega_0 t = e^{-at}\left[\frac{e^{+j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]$$

$$= \frac{1}{2}\left[e^{-(a-j\omega_0)t} + e^{-(a+j\omega_0)t}\right]$$

$$\Big\downarrow \mathcal{L}$$

$$\frac{1}{2}\left[\frac{1}{s+a-j\omega_0} + \frac{1}{s+a+j\omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+j\omega_0+s+a-j\omega_0}{[(s+a)-j\omega_0][(s+a)+j\omega_0]}\right] = \frac{(s+a)}{(s+a)^2+\omega_0^2}$$

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = ?$$

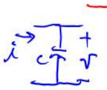
$$\int_0^\infty \frac{d}{dt}f(t)\, e^{-st}\,dt$$

$$= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)\, d(e^{-st})$$

Re $s > 0$

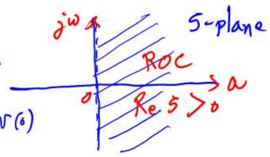
$$= 0 - f(0) - \int_0^\infty f(t)(-s)e^{-st}\,dt$$

$$= sF(s) - f(0)$$



$i = C\frac{dv}{dt}$

$I(s) = C\,sV(s) - v(0)$




Example

$$\sin\omega_0 t = -\frac{1}{\omega_0}\frac{d}{dt}(\cos\omega_0 t)$$

$$\frac{d}{dt}f(t) \rightarrow sF(s) - f(0)$$

$$\Big\downarrow \mathcal{L}$$

$$-\frac{1}{\omega_0}\left(s\left[\frac{s}{s^2+\omega_0^2}\right] - 1\right)$$

$$= -\frac{1}{\omega_0}\,\frac{s^2-(s^2+\omega_0^2)}{s^2+\omega_0^2} = \frac{\omega_0}{s^2+\omega_0^2}$$

same

Alternatively

$$\sin\omega_0 t = \frac{e^{+j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j}\left[e^{+j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$\xrightarrow{\mathcal{L}} \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right] = \frac{1}{2j}\left[\frac{s+j\omega_0-s+j\omega_0}{s^2+\omega_0^2}\right] = \frac{\omega_0}{s^2+\omega_0^2}$$