ECE103 Lecture 20, November 16, 2018

Exercise 3-2: Determine the Laplace transform of the causal sawtooth waveform shown in Fig. E3-2 (compare with Example 1-4).

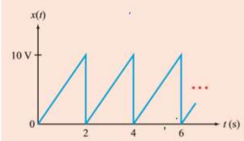


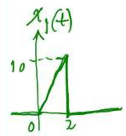
Figure E3-2

Answer:

$$\mathbf{X}(\mathbf{s}) = \mathbf{X}_1(\mathbf{s}) \sum_{n=0}^{\infty} e^{-2n\mathbf{s}} = \frac{\mathbf{X}_1(\mathbf{s})}{1 - e^{-2\mathbf{s}}},$$

where

$$\mathbf{X}_1(\mathbf{s}) = \int_0^2 (5t) e^{-\mathbf{s}t}\, dt = \frac{5}{\mathbf{s}^2}[1 - (2\mathbf{s}+1)e^{-2\mathbf{s}}].$$



$x(t) = x_1(t) + x_1(t-2) + \cdots + x_1(t-2n) \cdots = \sum_{n=0}^{\infty} x_1(t-2n)$

$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

when $x = e^{-2s}$

$1 + e^{-2s} + e^{-4s} + \cdots = 1/(1-e^{-2s})$

$$\mathcal{L}[x_1(t)] = \int_{-\infty}^{\infty} x_1(t) e^{-st} dt$$
$$= \int_0^2 5t\, e^{-st} dt$$
$$= \int_0^2 5t\, d\left(\frac{e^{-st}}{-s}\right)$$
$$= 5t\left(\frac{e^{-st}}{-s}\right)\Big|_0^2 - \int_0^2 \frac{e^{-st}}{-s} d(5t)$$
$$= 5(2)\frac{e^{-2s}}{-s} - 0 + \frac{5}{s}\int_0^2 e^{-st} dt$$
$$= -\frac{10}{s}e^{-2s} + \frac{5}{s}\underbrace{\left[\frac{e^{-st}}{-s}\right]_0^2}_{=\frac{1}{s}(1-e^{-2s})}$$
$$= \frac{5}{s^2}(1-e^{-2s}) - \frac{10}{s}e^{-2s}$$
$$= \frac{5}{s^2}(1-(2s+1)e^{-2s}) = X_1(s)$$

In general, identifying the locations of the poles and zeros of a function $\mathbf{X}(\mathbf{s})$ can be greatly facilitated if $\mathbf{X}(\mathbf{s})$ can be cast in the form

$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})} = \frac{A(\mathbf{s}-\mathbf{z}_1)(\mathbf{s}-\mathbf{z}_2)\ldots(\mathbf{s}-\mathbf{z}_m)}{(\mathbf{s}-\mathbf{p}_1)(\mathbf{s}-\mathbf{p}_2)\ldots(\mathbf{s}-\mathbf{p}_n)},$$

where the zeros $\mathbf{z}_1$ to $\mathbf{z}_m$ are the roots of $\mathbf{N}(\mathbf{s}) = 0$ and the poles $\mathbf{p}_1$ to $\mathbf{p}_n$ are the roots of $\mathbf{D}(\mathbf{s}) = 0$. As we will see in later chapters, the specific locations of poles and zeros in the s-plane carry great significance when designing frequency filters or characterizing their performance.

Occasionally, $\mathbf{X}(\mathbf{s})$ may have repeated poles or zeros, such as $\mathbf{z}_1 = \mathbf{z}_2$ or $\mathbf{p}_1 = \mathbf{p}_2$. Multiple zeros are marked by that many concentric circles, such as "◎" for two identical zeros, and multiple poles are marked by overlapping Xs "✖."

Concept Question 3-3: How does one determine the poles and zeros of a rational function $\mathbf{X}(\mathbf{s})$? (See S²)

Exercise 3-3: Determine the poles and zeros of $\mathbf{X}(\mathbf{s}) = (\mathbf{s}+a)/[(\mathbf{s}+a)^2 + \omega_0^2]$.

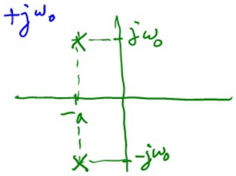
Answer: $\mathbf{z} = (-a + j0)$, $\mathbf{p}_1 = (-a - j\omega_0)$, and $\mathbf{p}_2 = (-a + j\omega_0)$. (See S²)

$$D(s) = (s+a)^2 + \omega_0^2 = 0$$
$$= (s+a)^2 - (j\omega_0)^2$$
$$= (s+a+j\omega_0)(s+a-j\omega_0)$$

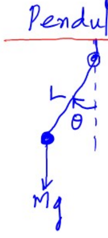
$a^2 - b^2 = (a+b)(a-b)$

$\Rightarrow p_1 = -a - j\omega_0$

$p_2 = -a + j\omega_0$



Pendulum Example (page 9 of the textbook)



$$ML\frac{d^2\theta(t)}{dt^2} = -Mg\sin\theta(t)$$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

for small θ, $\theta^3, \theta^5, \cdots \to 0$

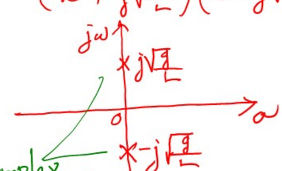
$$\Rightarrow \boxed{ML\frac{d^2\theta(t)}{dt^2} + Mg\,\theta(t) = 0}$$

$\downarrow \mathcal{L}$ with i.c = 0

$$ML s^2 \Theta(s) + Mg\,\Theta(s) = 0$$
$$\left(s^2 + \frac{g}{L}\right)\Theta(s) = 0$$

Characteristic eq.

$$s^2 + \frac{g}{L} = 0$$
$$s^2 - \left(j\sqrt{\frac{g}{L}}\right)^2 = 0$$
$$\left(s + j\sqrt{\frac{g}{L}}\right)\left(s - j\sqrt{\frac{g}{L}}\right) = 0$$



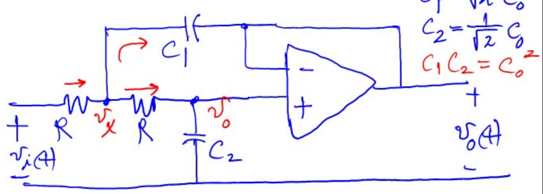
complex conjugate pair that leads to oscillation

Butterworth Filter

$C_1 = \sqrt{2}\,C_0$

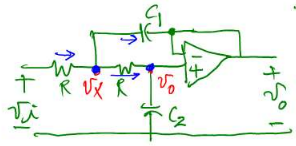
$C_2 = \frac{1}{\sqrt{2}}C_0$

$C_1 C_2 = C_0^2$



$$v_x = R\,i\left(= C_2\frac{dv_o}{dt}\right) + v_o$$
$$= RC_2\frac{dv_o}{dt} + v_o$$

Time Domain Analysis



$$v_x = R\left(C_2\frac{dv_o(t)}{dt}\right) + v_o$$

KCL at node x

$$\frac{v_i - v_x}{R} = C_1\frac{d(v_x - v_o)}{dt} + C_2\frac{dv_o}{dt} \quad \left(= C_1\frac{d}{dt}\left[RC_2\frac{dv_o}{dt}\right]\right)$$

$$v_i - v_x = R\left[RC_1C_2\frac{d^2v_o}{dt^2} + C_2\frac{dv_o}{dt}\right]$$
$$v_i = R^2C_1C_2\frac{d^2v_o}{dt^2} + RC_2\frac{dv_o}{dt} + v_x$$
$$= R^2C_1C_2\frac{d^2v_o}{dt^2} + RC_2\frac{dv_o}{dt} + \left(RC_2\frac{dv_o}{dt} + v_o\right)$$
$$\Rightarrow v_i = R^2C_1C_2\frac{d^2v_o}{dt^2} + 2RC_2\frac{dv_o}{dt} + v_o$$

For i.c. = 0 $\quad V_i(s) = \left(R^2C_1C_2 s^2 + 2RC_2 s + 1\right)V_o(s)$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{R^2C_1C_2\left(s^2 + \frac{2}{RC_1}s + \frac{1}{R^2C_1C_2}\right)}$$

$$H(s) = \frac{1}{(RC_0)^2\left(s^2 + \frac{1}{\sqrt{2}RC_0}s + \left(\frac{1}{RC_0}\right)^2\right)}$$

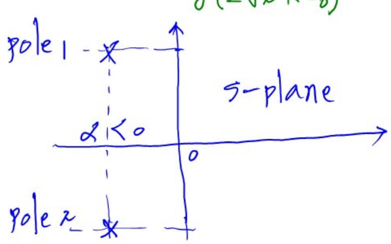
$C_1 = \sqrt{2}C_0$

$C_2 = \frac{1}{\sqrt{2}}C_0$

$$= \frac{N(s)}{D(s)}, \quad N(s) = \left(\frac{1}{RC_0}\right)^2$$

$$D(s) = s^2 + \frac{2}{2\sqrt{2}RC_0}s + \left(\frac{1}{RC_0}\right)^2$$
$$= \left(s + \frac{1}{2\sqrt{2}RC_0}\right)^2 + \frac{7}{8}\left(\frac{1}{RC_0}\right)^2$$
$$= \left(s + \frac{1}{2\sqrt{2}RC_0}\right)^2 - \left(j\frac{\sqrt{7}}{2\sqrt{2}RC_0}\right)^2$$

$$D(s) = 0 \Rightarrow s_{1,2} = -\left(\frac{1}{2\sqrt{2}\,RC_0}\right) \pm j\left(\frac{\sqrt{7}}{2\sqrt{2}RC_0}\right) = \alpha \pm j\beta$$



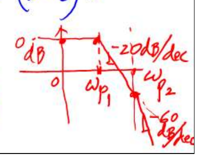
For Bode plot

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + 2RC_2 s + R^2C_1C_2 s^2}$$
$$= \frac{1}{1 + \frac{2}{\sqrt{2}}RC_0 s + (RC_0)^2 s^2}$$

$C_1 = \sqrt{2}C_0$

$C_2 = \frac{1}{\sqrt{2}}C_0$

$$H(\omega) = \frac{1}{\left(1 + j\frac{\omega}{\omega_{p1}}\right)\left(1 + j\frac{\omega}{\omega_{p2}}\right)}$$



How do we identify $\omega_{p_1}, \omega_{p_2}$?

$$\left(1 + j\frac{\omega}{\omega_{p_1}}\right)\left(1 + j\frac{\omega}{\omega_{p_2}}\right) = 1 + j\omega\frac{2}{\sqrt{2}}RC_0 + (j\omega RC_0)^2$$

$$= 1 + j\omega\left(\frac{1}{\omega_{p_1}} + \frac{1}{\omega_{p_2}}\right) + \frac{(j\omega)^2}{\omega_{p_1}\omega_{p_2}}$$

$$\Rightarrow \frac{1}{\omega_{p_1}} + \frac{1}{\omega_{p_2}} = \sqrt{2}\,RC_0 \quad (1)$$

$$\frac{1}{\omega_{p_1}\omega_{p_2}} = (RC_0)^2 \Rightarrow \frac{1}{\omega_{p_1}} = \omega_{p_2}(RC_0)^2 \quad (2)$$

$$(2)\to(1) \Rightarrow \omega_{p_2}(RC_0)^2 + \frac{1}{\omega_{p_2}} = \sqrt{2}\,RC_0$$

$$(\omega_{p_2})^2(RC_0)^2 + 1 = \sqrt{2}\,RC_0\,\omega_{p_2}$$

$$(RC_0)^2\omega_{p_2}^2 - \sqrt{2}\,RC_0\,\omega_{p_2} + 1 = 0 \rightarrow \text{Find } \omega_{p_2}$$

$$\omega_{p_2} = \frac{-(-\sqrt{2}\,RC_0) \pm \sqrt{(-2\sqrt{2}RC_0)^2 - 4(RC_0)^2\cdot 1}}{2(RC_0)^2}$$

$$= \frac{\sqrt{2} \pm \sqrt{8-4}}{2RC_0} = \frac{\sqrt{2} \pm 2}{2RC_0}$$

$$\Rightarrow \omega_{p_2} = \frac{\sqrt{2}+2}{2RC_0} = \underline{\left(\frac{2+\sqrt{2}}{2}\right)\frac{1}{RC_0}}$$

$$\omega_{p_1} = \frac{1}{\omega_{p_2}(RC_0)^2} = \frac{2RC_0}{(\sqrt{2}+2)(RC_0)^2}$$

$$= \underline{\left(\frac{2}{2+\sqrt{2}}\right)\frac{1}{RC_0}}$$

If $R = 0.1\,M\Omega$, $C_0 = 0.1\,\mu F$

$$\omega_{p_1} = \frac{2}{2+\sqrt{2}}100 = \underline{58.58}, \quad \omega_{p_2} = \frac{2+\sqrt{2}}{2}100 = \underline{220.7}$$

$$\cos[\omega_0(t-T)]\,u(t-T) \quad \longleftrightarrow \quad e^{-Ts}\frac{s}{s^2+\omega_0^2}. \quad (3.19)$$

Had we analyzed a linear circuit (or system) driven by a sinusoidal source that started at $t = 0$ and then wanted to reanalyze it anew, but we wanted to delay both the cosine function and the start time by T, Eq. (3.19) provides an expedient approach to obtaining the transform of the delayed cosine function.

Exercise 3-4: Determine, for $T \geq 0$,

$$\mathcal{L}\{[\sin\omega_0(t-T)]\,u(t-T)\}.$$

Answer: $e^{-Ts}\dfrac{\omega_0}{s^2+\omega_0^2}$. (See S2)

$\sin\omega_0(t-T)\;u(t-T) \rightarrow$

$\downarrow \mathcal{L}$

$$\int_0^\infty \sin\omega_0(t-T)\,u(t-T)\,e^{-st}dt$$

$$= \int_0^\infty \sin\omega_0(t-T)\,u(t-T)\,e^{-s(t-T)}dt \; e^{-sT}$$

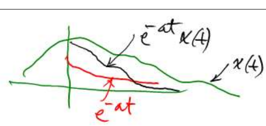
$$= e^{-sT}\int_0^\infty \sin\omega_0(t-T)\,u(t-T)\,e^{-s(t-T)}d(t-T)$$

$$\underset{\tau = t-T}{\overset{\uparrow}{=}} e^{-sT}\int_{-T}^\infty \sin\omega_0\tau\,u(\tau)\,e^{-s\tau}d\tau$$

$t = 0$, $\tau = -T$; ∞, ∞

$$= e^{-sT}\int_0^\infty \sin\omega_0\tau\,e^{-s\tau}d\tau$$

$$= e^{-sT}\mathcal{L}[\sin\omega_0 t] = \underline{e^{-sT}\frac{\omega_0}{s^2+\omega_0^2}}$$



3-3.3 Frequency Shift

According to the time-shift property, if t is replaced with $(t-T)$ in the time domain, $\mathbf{X}(s)$ gets multiplied by e^{-Ts} in the s-domain. Within a $(-)$ sign, the converse is also true: if s is replaced with $(s+a)$ in the s-domain, $x(t)$ gets multiplied by e^{-at} in the time domain. Thus,

$$e^{-at}\,x(t) \quad \longleftrightarrow \quad \mathbf{X}(s+a). \quad (3.20)$$

(frequency shift property)

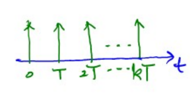
$$\int_0^\infty e^{-at}x(t)\,e^{-st}dt = \int_0^\infty x(t)\,e^{-\underbrace{(s+a)}_{=\hat{s}}t}dt$$

$$= X(\hat{s}) \underset{\hat{s}=s+a}{\overset{\uparrow}{=}} \underline{X(s+a)}$$

e.g. $\delta_T(t)\,u(t) = \sum_{k=0}^{\infty}\delta(t-kT)$

$$\mathcal{L}[\delta_T(t)\,u(t)] = \mathcal{L}\left[\sum_{k=0}^{\infty}\delta(t-kT)\right]$$

$$= \sum_{k=0}^{\infty}\mathcal{L}[\delta(t)]\,e^{-kTs}$$



$$= \sum_{k=0}^{\infty} e^{-kTs}$$

3-3.4 Time Differentiation

Differentiating $x(t)$ in the time domain is equivalent to (a) multiplying $\mathbf{X}(\mathbf{s})$ by $\mathbf{s}$ in the s-domain and then (b) subtracting $x(0^-)$ from $\mathbf{s}\,\mathbf{X}(\mathbf{s})$:

$$x' = \frac{dx}{dt} \leftrightarrow \mathbf{s}\,\mathbf{X}(\mathbf{s}) - x(0^-). \qquad (3.21)$$

(time-differentiation property)

To verify Eq. (3.21), we start with the standard definition for the Laplace transform:

$$\mathcal{L}[x'] = \int_{0^-}^{\infty} \frac{dx}{dt}\, e^{-\mathbf{s}t}\, dt. \qquad (3.22)$$

$$= x(t)e^{-st}\Big|_{0_-}^{\infty} - \int_{0_-}^{\infty} x(t)\, d\left(e^{-st}\right)$$

$$= 0 - x(0_-) - (-s)\underbrace{\int_{0_-}^{\infty} x(t) e^{-st} dt}_{= X(s)}$$

$$= -x(0_-) + s X(s)$$

Integration by parts with

$$u = e^{-\mathbf{s}t}, \qquad du = -\mathbf{s}e^{-\mathbf{s}t}\, dt,$$

$$dv = \left(\frac{dx}{dt}\right) dt, \qquad \text{and} \qquad v = x$$

gives

$$\mathcal{L}[x'] = uv\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\, du$$

$$= e^{-\mathbf{s}t}\, x(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} -\mathbf{s}\, x(t)\, e^{-\mathbf{s}t}\, dt$$

$$= -x(0^-) + \mathbf{s}\,\mathbf{X}(\mathbf{s}), \qquad (3.23)$$

which is equivalent to Eq. (3.21).

Higher derivatives can be obtained by repeating the application of Eq. (3.21). For the second derivative of $x(t)$, we have

$$x'' = \frac{d^2x}{dt^2} \leftrightarrow \mathbf{s}^2\,\mathbf{X}(\mathbf{s}) - \mathbf{s}\,x(0^-) - x'(0^-), \qquad (3.24)$$

(second-derivative property)

$$\frac{d^2}{dt^2}x(t) = \underbrace{\frac{d}{dt}\Big(\overbrace{\frac{dx(t)}{dt}}^{= f(t)}\Big)}$$

$$\xrightarrow{\mathcal{L}} \quad s X(s) - x(0_-) = F(s)$$

$$\xrightarrow{\mathcal{L}} \quad s\big[\overbrace{s X(s) - x(0_-)}^{= F(s)}\big] - \overbrace{\frac{dx}{dt}(0_-)}^{= f(0_-)}$$

$$= s^2 X(s) - s\,x(0_-) - x'(0_-)$$

3-3.5 Time Integration

Integration of $x(t)$ in the time domain is equivalent to dividing $\mathbf{X}(\mathbf{s})$ by $\mathbf{s}$ in the s-domain:

$$\int_0^t x(t')\, dt' \leftrightarrow \frac{1}{\mathbf{s}}\,\mathbf{X}(\mathbf{s}). \qquad (3.25)$$

(time-integration property)

Application of the Laplace transform definition gives

$$\mathcal{L}\left[\int_{0^-}^{t} x(t')\, dt'\right] = \int_{0^-}^{\infty} \underbrace{\left[\int_{0^-}^{t} x(t')\, dt'\right]}_{u} \underbrace{e^{-\mathbf{s}t}\, dt}_{dv}. \qquad (3.26)$$

Integration by parts with

$$u = \int_{0^-}^{t} x(t')\, dt', \qquad du = x(t)\, dt,$$

$$dv = e^{-\mathbf{s}t}\, dt, \qquad \text{and} \qquad v = -\frac{e^{-\mathbf{s}t}}{\mathbf{s}}$$

leads to

$$\mathcal{L}\left[\int_{0^-}^{t} x(t')\, dt'\right]$$

$$= uv\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\, du$$

$$= \underbrace{\left[-\frac{e^{-\mathbf{s}t}}{\mathbf{s}} \int_{0^-}^{t} x(t')\, dt'\right]\Bigg|_{0^-}^{\infty}}_{= 0 - 0 = 0} + \frac{1}{\mathbf{s}} \int_{0^-}^{\infty} x(t)\, e^{-\mathbf{s}t}\, dt = \frac{1}{\mathbf{s}}\,\mathbf{X}(\mathbf{s}). \qquad (3.27)$$

Both limits on the first term on the right-hand side yield zero values.

For example, since

$$\delta(t) \leftrightarrow 1,$$

it follows that

$$u(t) = \int_{0^-}^{t} \delta(t')\, dt' \leftrightarrow \frac{1}{\mathbf{s}}$$

and

$$r(t) = \int_{0^-}^{t} u(t')\, dt' \leftrightarrow \frac{1}{\mathbf{s}^2}.$$

Application

$$\cos\omega_0 t \xrightarrow{\mathcal{L}} \frac{s}{s^2+\omega_0^2}$$

$$\frac{d}{dt}(\cos\omega_0 t) = -\omega_0 \sin\omega_0 t$$

$$\sin\omega_0 t = -\frac{1}{\omega_0}\frac{d}{dt}(\cos\omega_0 t)$$

$$\downarrow\mathcal{L} \qquad \downarrow\mathcal{L}$$

$$= -\frac{1}{\omega_0}\Big[s\Big(\frac{s}{s^2+\omega_0^2}\Big) - \underbrace{\cos\omega_0(0_-)}_{=1}\Big]$$

$$= -\frac{1}{\omega_0}\Big[\frac{s^2}{s^2+\omega_0^2} - 1\Big]$$

$$= -\frac{1}{\omega_0}\,\frac{\cancel{s^2} - (\cancel{s^2}+\omega_0^2)}{s^2+\omega_0^2} = \frac{\omega_0}{s^2+\omega_0^2}$$

$$= \mathcal{L}[\sin\omega_0 t]$$

3-3.6 Initial- and Final-Value Theorems

The relationship between $x(t)$ and $\mathbf{X}(\mathbf{s})$ is such that the initial value $x(0^+)$ and the final value $x(\infty)$ of $x(t)$ can be determined directly from the expression of $\mathbf{X}(\mathbf{s})$—provided certain conditions are satisfied (as discussed later in this subsection).

Consider the derivative property represented by Eq. (3.23) as

$$\mathcal{L}[x'] = \int_{0^-}^{\infty} \frac{dx}{dt}\, e^{-\mathbf{s}t}\, dt = \mathbf{s}\,\mathbf{X}(\mathbf{s}) - x(0^-). \qquad (3.28)$$

If we take the limit as $\mathbf{s} \to \infty$ while recognizing that $x(0^-)$ is independent of $\mathbf{s}$, we get

$$\lim_{\mathbf{s}\to\infty}\left[\int_{0^-}^{\infty} \frac{dx}{dt}\, e^{-\mathbf{s}t}\, dt\right] = \lim_{\mathbf{s}\to\infty}[\mathbf{s}\,\mathbf{X}(\mathbf{s})] - x(0^-). \qquad (3.29)$$

The integral on the left-hand side can be split into two integrals: one over the time segment $(0^-, 0^+)$, for which $e^{-st} = 1$, and another over the segment $(0^+, \infty)$. Thus,

$$\lim_{s\to\infty}\left[\int_{0^-}^{\infty}\frac{dx}{dt}\,e^{-st}\,dt\right]$$

$$= \lim_{s\to\infty}\left[\int_{0^-}^{0^+}\frac{dx}{dt}\,dt + \int_{0^+}^{\infty}\frac{dx}{dt}\,e^{-st}\,dt\right] = x(0^+) - x(0^-). \quad (3.30)$$

As $s \to \infty$, the exponential function e^{-st} causes the integrand of the last term to vanish. Equating Eqs. (3.29) and (3.30) leads to

$$x(0^+) = \lim_{s\to\infty} s\,\mathbf{X}(s), \quad (3.31)$$

(initial-value theorem)

which is known as the ***initial-value theorem***.

A similar treatment in which s is made to approach 0 (instead of ∞) in Eq. (3.29) leads to the ***final-value theorem***:

$$x(\infty) = \lim_{s\to 0} s\,\mathbf{X}(s). \quad (3.32)$$

(final-value theorem)

We should note that Eq. (3.32) is useful for determining $x(\infty)$, *so long as $x(\infty)$ exists*. Otherwise, application of Eq. (3.32) may lead to an erroneous result. Consider, for example, $x(t) = \cos(\omega_0 t)\,u(t)$, which does not have a unique value as $t \to \infty$. Yet, application of Eq. (3.32) to Eq. (3.9) leads to $x(\infty) = 0$, which is incorrect.

---

e.g. $x(t) = e^{-3t}\cos 10t \xrightarrow{\mathcal{L}} \dfrac{(s+3)}{(s+3)^2+10^2}$

$x(\infty) = \lim_{s\to 0} s\mathbf{X}(s) = \lim_{s\to 0} \dfrac{s(s+3)}{(s+3)^2+10^2} = 0$

Indeed $x(t) = e^{-3t}\cos 3t \to 0$ as $t \to \infty$

Inversely $x(0_+) = \lim_{s\to\infty} s\mathbf{X}(s)$

$= \lim_{s\to\infty} \dfrac{s(s+3)}{(s+3)^2+10^2} = \lim_{s\to\infty} \dfrac{1+\frac{3}{s^2}}{(1+\frac{3}{s})^2+\frac{10^2}{s^2}}$

$= \dfrac{1}{1} = 1$

Indeed $x(t=0_+) = e^{-3t}\cos 10t\,\big|_{t=0_+} = 1\cdot 1 = 1$

Example 3-4: Initial and Final Values

Determine the initial and final values of a function $x(t)$ whose Laplace transform is given by

$$\mathbf{X}(s) = \frac{25s(s+3)}{(s+1)(s^2+2s+36)}.$$

Solution: Application of Eq. (3.31) gives

$$x(0^+) = \lim_{s\to\infty} s\,\mathbf{X}(s) = \lim_{s\to\infty}\frac{25s^2(s+3)}{(s+1)(s^2+2s+36)}.$$

To avoid the problem of dealing with ∞, it is often more convenient to first apply the substitution $s = 1/\mathbf{u}$, rearrange the function, and then find the limit as $\mathbf{u} \to 0$. That is,

$$x(0^+) = \lim_{\mathbf{u}\to 0}\frac{25(1/\mathbf{u}^2)(1/\mathbf{u}+3)}{(1/\mathbf{u}+1)(1/\mathbf{u}^2+2/\mathbf{u}+36)}$$

$$= \lim_{\mathbf{u}\to 0}\frac{25(1+3\mathbf{u})}{(1+\mathbf{u})(1+2\mathbf{u}+36\mathbf{u}^2)}$$

$$= \frac{25(1+0)}{(1+0)(1+0+0)} = 25.$$

$\leftrightarrow$ $s X(s) = \dfrac{s\,25s(s+3)}{(s+1)(s^2+2s+36)}$

$\lim_{s\to\infty} s X(s) = 25$

To determine $x(\infty)$, we apply Eq. (3.32):

$$x(\infty) = \lim_{s\to 0} s\,\mathbf{X}(s) = \lim_{s\to 0}\frac{25s^2(s+3)}{(s+1)(s^2+2s+36)} = 0.$$

Exercise 3-6: Determine the initial and final values of $x(t)$ if its Laplace transform is given by

$$\mathbf{X}(s) = \frac{s^2+6s+18}{s(s+3)^2}.$$

$\lim_{s\to\infty} sX(s)$

Answer: $x(0^+) = 1$, $x(\infty) = 2$. (See S2)

$\lim_{s\to 0} sX(s)$

3-3.7 Frequency Differentiation

Given the definition of the Laplace transform, namely,

$$\mathbf{X}(s) = \mathcal{L}[x(t)] = \int_{0^-}^{\infty} x(t)\,e^{-st}\,dt, \quad (3.33)$$

if we take the derivative with respect to s on both sides, we have

$$\frac{d\,\mathbf{X}(s)}{ds} = \int_{0^-}^{\infty}\frac{d}{ds}[x(t)\,e^{-st}]\,dt$$

$$= \int_{0^-}^{\infty}[-t\,x(t)]e^{-st}\,dt = \mathcal{L}[-t\,x(t)], \quad (3.34)$$

where we recognize the integral as the Laplace transform of the function $[-t\,x(t)]$. Rearranging Eq. (3.34) provides the ***frequency differentiation relation***:

$$t\,x(t) \quad \longleftrightarrow \quad -\frac{d\,\mathbf{X}(s)}{ds} = -\mathbf{X}'(s), \quad (3.35)$$

(frequency differentiation property)

which states that multiplication of $x(t)$ by $-t$ in the time domain is equivalent to differentiating $\mathbf{X}(s)$ in the s-domain.

---

e.g.) $t\,x(t) = t \xrightarrow{\mathcal{L}} \dfrac{1}{s^2}$

$x(t) = u(t)$

$\mathbf{X}(s) = \dfrac{1}{s}$

$-\dfrac{d}{ds}\mathbf{X}(s) = -\dfrac{d}{ds}\left(\dfrac{1}{s}\right) = -\left(-\dfrac{1}{s^2}\right) = \dfrac{1}{s^2}$

e.g.) $t\,r(t) = t^2 \xrightarrow{\mathcal{L}}$

$R(s) = \mathcal{L}[r(t)] = \dfrac{1}{s^2}$, $-\dfrac{d}{ds}R(s) = -\left(-\dfrac{2s}{(s^2)^2}\right) = \dfrac{2}{s^3}$

Let us do direct $\mathcal{L}$:

$\mathcal{L}[t^2] = \int_0^\infty t^2 e^{-st}dt = \int_0^\infty t^2\,d\left(\dfrac{e^{-st}}{-s}\right) = t^2\left(\dfrac{e^{-st}}{-s}\right)\Big|_0^\infty - \int\left(-\dfrac{e^{-st}}{s}\right)2t\,dt$

$= \dfrac{2}{s}\mathcal{L}[t] = \dfrac{2}{s}\cdot\dfrac{1}{s^2} = \dfrac{2}{s^3}$

Example 3-5: Applying the Frequency Differentiation Property

Given that

$$\mathbf{X}(\mathbf{s}) = \mathcal{L}[e^{-at}\, u(t)] = \frac{1}{\mathbf{s}+a}\,,$$

apply Eq. (3.35) to obtain the Laplace transform of $t e^{-at}\, u(t)$.

Solution:

$$\mathcal{L}[t e^{-at}\, u(t)] = -\frac{d}{d\mathbf{s}}\,\mathbf{X}(\mathbf{s}) = -\frac{d}{d\mathbf{s}}\left[\frac{1}{\mathbf{s}+a}\right] = \frac{1}{(\mathbf{s}+a)^2}\,.$$

3-3.8 Frequency Integration

Integrating both sides of Eq. (3.33) from s to ∞ gives

$$\int_{\mathbf{s}}^{\infty} \mathbf{X}(\mathbf{s}')\, d\mathbf{s}' = \int_{\mathbf{s}}^{\infty}\left[\int_{0^-}^{\infty} x(t)\, e^{-\mathbf{s}'t}\, dt\right] d\mathbf{s}'. \qquad (3.36)$$

Since t and $\mathbf{s}'$ are independent variables, we can interchange the order of the integration on the right-hand side of Eq. (3.36),

$$\begin{aligned}\int_{\mathbf{s}}^{\infty} \mathbf{X}(\mathbf{s}')\, d\mathbf{s}' &= \int_{0^-}^{\infty}\left[\int_{\mathbf{s}}^{\infty} x(t)\, e^{-\mathbf{s}'t}\, d\mathbf{s}'\right] dt \\ &= \int_{0^-}^{\infty}\left[\frac{x(t)}{-t}\, e^{-\mathbf{s}'t}\Big|_{\mathbf{s}}^{\infty}\right] dt \\ &= \int_{0^-}^{\infty}\left[\frac{x(t)}{t}\right] e^{-\mathbf{s}t}\, dt = \mathcal{L}\left[\frac{x(t)}{t}\right]. \qquad (3.37)\end{aligned}$$

This ***frequency integration property*** can be expressed as

$$\frac{x(t)}{t} \quad \longleftrightarrow \quad \int_{\mathbf{s}}^{\infty} \mathbf{X}(\mathbf{s}')\, d\mathbf{s}'. \qquad (3.38)$$

(frequency integration property)

Table 3-1: Properties of the Laplace transform for causal functions; i.e., $x(t) = 0$ for $t < 0$.

Property	$x(t)$		$\mathbf{X}(\mathbf{s}) = \mathcal{L}[x(t)]$
1. Multiplication by constant	$K\, x(t)$	$\longleftrightarrow$	$K\, \mathbf{X}(\mathbf{s})$
2. Linearity	$K_1\, x_1(t) + K_2\, x_2(t)$	$\longleftrightarrow$	$K_1\, \mathbf{X}_1(\mathbf{s}) + K_2\, \mathbf{X}_2(\mathbf{s})$
3. Time scaling	$x(at), \quad a > 0$	$\longleftrightarrow$	$\frac{1}{a}\mathbf{X}\left(\frac{\mathbf{s}}{a}\right)$
4. Time shift	$x(t-T)\, u(t-T)$	$\longleftrightarrow$	$e^{-T\mathbf{s}}\, \mathbf{X}(\mathbf{s})$
5. Frequency shift	$e^{-\mathbf{a}t}\, x(t)$	$\longleftrightarrow$	$\mathbf{X}(\mathbf{s}+\mathbf{a})$
6. Time 1st derivative	$x' = \frac{dx}{dt}$	$\longleftrightarrow$	$\mathbf{s}\, \mathbf{X}(\mathbf{s}) - x(0^-)$
7. Time 2nd derivative	$x'' = \frac{d^2x}{dt^2}$	$\longleftrightarrow$	$\mathbf{s}^2\mathbf{X}(\mathbf{s}) - \mathbf{s}x(0^-) - x'(0^-)$
8. Time integral	$\int_{0^-}^{t} x(t')\, dt'$	$\longleftrightarrow$	$\frac{1}{\mathbf{s}}\, \mathbf{X}(\mathbf{s})$
9. Frequency derivative	$t\, x(t)$	$\longleftrightarrow$	$-\frac{d}{d\mathbf{s}}\, \mathbf{X}(\mathbf{s}) = -\mathbf{X}'(\mathbf{s})$
10. Frequency integral	$\frac{x(t)}{t}$	$\longleftrightarrow$	$\int_{\mathbf{s}}^{\infty} \mathbf{X}(\mathbf{s}')\, d\mathbf{s}'$
11. Initial value	$x(0^+)$	$=$	$\lim_{\mathbf{s}\to\infty} \mathbf{s}\, \mathbf{X}(\mathbf{s})$
12. Final value	$\lim_{t\to\infty} x(t) = x(\infty)$	$=$	$\lim_{\mathbf{s}\to 0} \mathbf{s}\, \mathbf{X}(\mathbf{s})$
13. Convolution	$x_1(t) * x_2(t)$	$\longleftrightarrow$	$\mathbf{X}_1(\mathbf{s})\, \mathbf{X}_2(\mathbf{s})$