Exercise 3.2: Determine the Laplace transform of the input waveform shown in Fig. 3.2-1 (compare with Example 1.4).

\[ X_1(t) = 10 \sin(5t) \]

\[ X_2(t) = 10 \cos(5t) \]

\[ X_3(t) = 10 \sin(5t) + 5 \cos(5t) \]

\[ X_4(t) = 10 \sin(5t) - 5 \cos(5t) \]

\[ X_5(t) = 10 \sin(5t) + 5 \cos(5t) \]

\[ X_6(t) = 10 \sin(5t) - 5 \cos(5t) \]

\[ X_7(t) = 10 \sin(5t) + 5 \cos(5t) \]

\[ X_8(t) = 10 \sin(5t) - 5 \cos(5t) \]

In general, identifying the locations of the poles and zeros of a function \( X(s) \) can be greatly facilitated if \( X(s) \) can be cast in the form

\[ X(s) = \frac{N(s)}{D(s)} \]

where the zeros \( z_1 \) to \( z_m \) are the roots of \( N(s) = 0 \) and the poles \( p_1 \) to \( p_n \) are the roots of \( D(s) = 0 \). As we will see in later chapters, the specific locations of poles and zeros in the s-plane carry great significance when designing frequency filters or characterizing their performance.

Occasionally, \( X(s) \) may have repeated poles or zeros, such as \( z_1 = z_2 \) or \( p_1 = p_2 \). Multiple zeros are marked by that many concentric circles, such as \( \odot \) for two identical zeros, and multiple poles are marked by overlapping \( \odot \).

Concept Question 3.3: How does one determine the poles and zeros of a rational function \( X(s) \)? (See Fig. 3.3)

Exercise 3.3: Determine the poles and zeros of \( X(s) = (s^2 + a) / (s^3 + a^2 + a + b) \).

Answer: \( z = (-a \pm jb) \), \( p_1 = (-a - jb) \), and \( p_2 = (-a + jb) \). (See Fig. 3.3)

Pendulum Example (from page 4 of the textbook)

\[ M L \ddot{\theta} + M g L \dot{\theta} \sin \theta = 0 \]

\[ \sin \theta = 0 - \frac{g}{L} \theta \]

\[ \text{for small } \theta, \theta \approx \frac{g}{L} \theta \]

\[ \Rightarrow M L \ddot{\theta} + M g L \dot{\theta} \sin \theta = 0 \]

\[ \ddot{\theta} = 0 \]

\[ \theta = \frac{g}{L} \theta \]

\[ \text{characteristic eq.} \]

\[ \lambda^2 + \frac{g}{L} \lambda = 0 \]

\[ \lambda^2 - (\alpha + j\beta) \lambda = 0 \]

\[ \text{complex conjugate pair that leads to oscillation} \]
Butterworth Filter

\[ V_o = R \left( C_1 \frac{dV_1}{dt} + V_1 \right) = R C_1 \frac{dV_1}{dt} + V_1 \]

Time Domain Analysis

\[ V_x = R \left( C_2 \frac{dV_2}{dt} \right) + V_0 \]

KCL at node X:

\[ \frac{V_x - V_C}{R} = C_1 \frac{d(V_x - V_C)}{dt} + \left( C_1 \frac{dV_1}{dt} \right) \]

\[ V_1 - V_C = R \left[ R C_1 C_2 \frac{dV_0}{dt} + C_2 \frac{dV_2}{dt} \right] \]

\[ V_2 = R C_1 C_2 \frac{dV_0}{dt} + R C_2 \frac{dV_2}{dt} + V_x \]

\[ V_x = V_C \left( 1 + 2 R C_2 \frac{dV_0}{dt} \right) \]

\[ H(s) = \frac{1}{\left( R C_0 \right)^2 s^2 + \frac{1}{2 \sqrt{2}} R C_0 s + \left( \frac{1}{R C_0} \right)} \]

\[ C_1 = \frac{1}{2 \sqrt{2}} C_0 \]

\[ C_2 = \frac{1}{2} \sqrt{2} C_0 \]

For Bode plot

\[ H(s) = \frac{V(s)}{V_i(s)} = \frac{1}{1 + 2 R C_2 s + R C_1 C_2 s^2} \]

\[ H(\omega) = \frac{1}{(1 + \frac{\omega^2}{\omega_n^2})(1 + \frac{2 \zeta \omega_n}{\omega_n})} \]
How do we identify \( y_1, y_2 \) and \( y_3 \)?

\[
(1 + \frac{j\omega}{\omega_p})(1 + \frac{j\omega}{\omega_C}) = 1 + \frac{j\omega}{\omega_p} \cdot RC_0 + \frac{j\omega}{\omega C} = 1 + \frac{j\omega}{\omega_p} + \frac{j\omega}{\omega_C} - \frac{j\omega}{\omega_p} \cdot \frac{j\omega}{\omega_C}
\]

\[
\Rightarrow y_1 + y_2 = \sqrt{EC} \quad (1)
\]

\[
y_1^2 + y_2^2 = RC_0 \quad (3)
\]

\[
(2) \Rightarrow y_1 y_2 = \sqrt{EC} \quad (5)
\]

\[
\text{Ans: } y_1 = \frac{\sqrt{EC}}{2}, y_2 = \frac{\sqrt{EC}}{2}
\]

If \( x(t) \) is multiplied by \( e^{-t\alpha} \), the \( \mathcal{L}\)-transform becomes

\[
\mathcal{L}(x(t)) \Rightarrow \mathcal{L}(e^{-t\alpha}x(t)) = \frac{X(s)}{s + \alpha}
\]

S-Transforms

\[
\mathcal{S}(x(t)) = \int_{0}^{\infty} x(t) \delta(t-T) \, dt
\]

\[
\mathcal{S}(x(t)) = e^{sT} \mathcal{S}(x(t - T))
\]

Frequency Shift

\[
\mathcal{L}[e^{sT}x(t)] = \mathcal{L}[x(t - T)]
\]

\[
\mathcal{L}[e^{sT}x(t)] = \mathcal{L}[x(t - T)] = \mathcal{L}[x(t) - T] = \frac{X(s)}{s + \alpha}
\]
5.3.4 Time Differentiation

Integration of \( x(t) \) in the time-domain is equivalent to dividing \( X(s) \) by \( s \) in the s-domain:

\[
\mathcal{L}\left\{ \int_0^t x(t') \, dt' \right\} = \frac{1}{s} X(s).
\]

(3.25)

Application of the Laplace transform definition gives

\[
\mathcal{L}\left\{ \int_0^t x(t') \, dt' \right\} = \int_0^\infty e^{-st} x(t') \, dt'.
\]

(3.26)

Integration by parts with

\[
\begin{align*}
\phi &= \int_0^t x(t') \, dt', \\
\psi &= x(t), \\
\phi &= e^{-st} \, dt, \quad \text{and} \quad \psi = \frac{e^{-st}}{s}.
\end{align*}
\]

(3.27)

Both limits on the first term on the right-hand side yield zero values. For example, since

\[
x(0) = 1,
\]

it follows that

\[
\begin{align*}
\phi(0) &= \int_0^0 x(t') \, dt' = 0, \\
\psi(0) &= \frac{1}{s}.
\end{align*}
\]

and

\[
\begin{align*}
\phi(\infty) &= \int_\infty^\infty e^{-st} x(t') \, dt' = 0, \\
\psi(\infty) &= \frac{1}{s}.
\end{align*}
\]

5.3.5 Time Integration

5.3.6 Initial- and Final-Value Theorems

The relationship between \( x(t) \) and \( X(s) \) is such that the initial value \( x(0^-) \) and the final value \( x(\infty) \) of \( x(t) \) can be determined directly from the expression of \( X(s) \)—provided certain conditions are satisfied (as discussed later in this subsection).

Consider the derivative property represented by Eq. (3.23) as

\[
\mathcal{L}\left\{ \int_0^t x(t) \, dt \right\} = sX(s) - x(0^-).
\]

(3.28)

If we take the limit as \( s \to \infty \) while recognizing that \( x(t) \) is independent of \( s \), we get

\[
\lim_{s \to \infty} \left[ \int_0^t e^{-st} \, dt \right] = \lim_{s \to \infty} [sX(s) - x(0^-)].
\]

(3.29)
The integral of the 1st-kind table can be split into two integrals:
\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx. \]
If \( c = \infty \), the upper limit of the integral is infinite.
\[ \lim_{c \to \infty} \int_a^c f(x) \, dx \]
which is known as the initial value theorem.
A similar theorem in which \( b \) is made to approach \( -\infty \) instead of \( c \) in Eq. (3.31), leads to the final value theorem.
\[ \lim_{b \to -\infty} x(b) = \lim_{b \to -\infty} X(s). \]
We should note that Eq. (3.32) is useful for determining \( x(\infty) \) or \( x(b) \) when \( s \to 0 \). Alternation of Eq. (3.32) may lead to an incorrect result. Consider, for example \( x(t) = \cos(t) \), which does not have a unique value \( x(\infty) = \lim_{t \to 0} \cos(t) \).
Application of Eq. (3.32) in Eq. (3.33) leads to \( x(\infty) = 0 \), which is incorrect.

**Example 3-4: Initial and Final Values**

Determine the initial and final values of a function \( x(t) \) whose Laplace transform is given by
\[ X(s) = \frac{2s^2(s+3)}{(s+1)(s+2)s+3}. \]

**Solution:** Application of Eq. (3.31) gives
\[ x(0^+) = \lim_{s \to \infty} s X(s) = \lim_{s \to \infty} \frac{2s^3(s+3)}{(s+1)(s+2)s+3}. \]

To avoid the problem of dealing with \( \infty \), it is often more convenient to first apply the substitution \( s = 1/\alpha \) and then find the limit as \( \alpha \to 0 \). That is,
\[ x(\alpha) = \lim_{\alpha \to 0} \frac{2s^3(s+3)}{(s+1)(s+2)s+3} \]
which, for example, evaluates as \( x(0^+) = \frac{2s^2(s+3)}{(s+1)(s+2)s+3} \).

To determine \( x(\infty) \), we apply Eq. (3.32): \( x(\infty) = \lim_{s \to 0} s X(s) = \lim_{s \to 0} \frac{2s^3(s+3)}{(s+1)(s+2)s+3} = 0. \)

**Exercise 3-4:** Determine the initial and final values of \( x(t) \) if its Laplace transform is given by
\[ X(s) = \frac{2s^2 + 6s + 18}{s(s+3)^2}. \]

**Answer:** \( x(0^+) = 1, x(\infty) = 2. \)

**3.3 Frequency Differentiation**

Given the definition of the Laplace transform, namely,
\[ X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t) e^{-st} \, dt, \quad \text{(3.33)} \]
if we take the derivative with respect to \( s \) on both sides, we have
\[ \frac{dX(s)}{ds} = \int_0^\infty \frac{d}{ds} x(t) e^{-st} \, dt \]
\[ = \int_0^\infty -t x(t) e^{-st} \, dt = -\mathcal{L}\{tx(t)\}. \quad \text{(3.34)} \]
where we recognize the integral as the Laplace transform of the function \( t x(t) \). Rearranging Eq. (3.34) provides the frequency differentiation relation:
\[ t x(t) = -\frac{d}{ds} X(s). \quad \text{(3.35)} \]
which states that multiplication of \( t x(t) \) by \( -1 \) in the time domain is equivalent to differentiating \( x(t) \) in the s-domain.
Example 3.6: Applying the Frequency Differentiation Property

Given that

$$X(s) = \mathcal{L}(e^{-at} u(t)) = \frac{1}{s + a}$$

apply Eq. (3.35) to obtain the Laplace transform of $e^{-at} u(t)$.

Solution:

$$\mathcal{L}(e^{-at} u(t)) = \frac{d}{ds} X(s) = -\frac{d}{ds} \left( \frac{1}{s + a} \right) = \frac{1}{(s + a)^2}$$

3.3.8 Frequency Integration

Integrating both sides of Eq. (3.35) from $s$ to $\infty$ gives

$$\int_{s}^{\infty} X'(s) ds = \int_{s}^{\infty} \left[ \frac{1}{s + a} e^{-a} ds \right] ds.$$  \hspace{1cm} (3.36)

Since $s$ and $s'$ are independent variables, we can interchange the order of the integration on the right-hand side of Eq. (3.36),

$$\int_{s}^{\infty} X'(s) ds = \int_{s}^{\infty} \left[ \frac{1}{s + a} e^{-a} ds \right] ds$$

$$= \int_{s}^{\infty} \left[ \frac{1}{s + a} e^{-a} ds \right] ds$$

This frequency integration property can be expressed as

$$\frac{2\pi i}{s} \rightarrow \frac{2\pi i}{s}$$

(frequency integration property)  \hspace{1cm} (3.37)

---

Table 3.1: Properties of the Laplace transform for causal functions $x(t), x(t)$, $t > 0$.

<table>
<thead>
<tr>
<th>Property</th>
<th>$X(s)$</th>
<th>$X(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Multiplication by constant</td>
<td>$k x(t)$</td>
<td>$k X(s)$</td>
</tr>
<tr>
<td>2. Linearity</td>
<td>$k_1 x(t) + k_2 x(t)$</td>
<td>$k_1 X(s) + k_2 X(s)$</td>
</tr>
<tr>
<td>3. Time scaling</td>
<td>$x(at)$, $a &gt; 0$</td>
<td>$\frac{1}{a} X\left(\frac{s}{a}\right)$</td>
</tr>
<tr>
<td>4. Time shift</td>
<td>$x(t - T)$</td>
<td>$e^{-sT}X(s)$</td>
</tr>
<tr>
<td>5. Frequency shift</td>
<td>$e^{st} x(t)$</td>
<td>$X(s + a)$</td>
</tr>
<tr>
<td>6. Time 1st derivative</td>
<td>$\frac{d}{dt} x(t)$</td>
<td>$s X(s) - x(0)$</td>
</tr>
<tr>
<td>7. Time 2nd derivative</td>
<td>$\frac{d^2}{dt^2} x(t)$</td>
<td>$s^2 X(s) - sx(0) - x(t)$</td>
</tr>
<tr>
<td>8. Time integral</td>
<td>$\int_{0}^{t} x(t') dt'$</td>
<td>$\frac{1}{s} X(s)$</td>
</tr>
<tr>
<td>9. Frequency derivative</td>
<td>$(s + a) x(t)$</td>
<td>$\frac{d}{ds} X(s)$</td>
</tr>
<tr>
<td>10. Frequency integral</td>
<td>$\frac{x(t)}{s}$</td>
<td>$\frac{1}{s} X(s)$</td>
</tr>
<tr>
<td>11. Initial value</td>
<td>$x(0)$</td>
<td>$X(0)$</td>
</tr>
<tr>
<td>12. Final value</td>
<td>$\lim_{t \to \infty} x(t)$</td>
<td>$\frac{1}{s} X(s)$</td>
</tr>
<tr>
<td>13. Convolution</td>
<td>$x_1(t) * x_2(t)$</td>
<td>$X_1(s) X_2(s)$</td>
</tr>
</tbody>
</table>