3-3.3 Time Integration

Integration of \( x(t) \) in the time-domain is equivalent to dividing \( X(s) \) by \( s \) in the s-domain:

\[
\mathcal{L} \left\{ \int_0^t x(t') \, dt' \right\} = \frac{1}{s} X(s).
\]

(time-integration property)

Application of the Laplace transform definition gives

\[
\mathcal{L} \left\{ \int_0^t x(t') \, dt' \right\} = \frac{1}{s} \int_0^s X(s') \, ds'.
\]

Integration by parts with

\[
\begin{align*}
\alpha &= \int_0^s x(t') \, dt', \\
\beta &= x(t) \, dt, \\
\gamma &= e^{-\alpha s} \, dt, \quad \text{and} \quad \delta = -\frac{e^{-\alpha s}}{s}
\end{align*}
\]

leads to

\[
\mathcal{L} \left\{ \int_0^t x(t') \, dt' \right\} = -\frac{1}{s^2} \left[ x(0^+) + \frac{1}{s} \int_0^s x(s') \, ds' \right] + \frac{1}{s} X(s),
\]

Both limits on the first term on the right-hand side yield zero values. For example, since

\[
\lim_{a \to 0} x(0^+) = 0,
\]

it follows that

\[
\mathcal{L} \left\{ \int_0^t x(t') \, dt' \right\} = \frac{1}{s} X(s).
\]

and

\[
\mathcal{L} \left\{ \int_0^t x(t') \, dt' \right\} = \frac{1}{s^2}.
\]

3-3.6 Initial- and Final-Value Theorems

The relationship between \( x(t) \) and \( X(s) \) is such that the initial value \( x(0^+) \) and the final value \( x(\infty) \) of \( x(t) \) can be determined directly from the expression of \( X(s) \)—provided certain conditions are satisfied (as discussed later in this subsection).

Consider the derivative property represented by Eq. (3.23) as

\[
\mathcal{L} \left\{ \frac{dx}{dt} \right\} = s X(s) - x(0^+),
\]

If we take the limit as \( s \to \infty \) while recognizing that \( x(0^+) \) is independent of \( s \), we get

\[
\lim_{s \to \infty} \left[ \int_0^\infty x(t') e^{-st'} \, dt' \right] = \lim_{s \to \infty} [s X(s) - x(0^+)].
\]
The integral on the left-hand side can be split into two integrals:

\[ \int_0^\infty e^{-sx} \, dx = \frac{1}{s} \left[ \frac{e^{-sx}}{-s} \right]_0^\infty = \frac{1}{s^2} \]

\[ \int_0^\infty e^{-sx} \, dx = \lim_{b \to \infty} \left[ \frac{e^{-sx}}{-s} \right]_0^b = \frac{1}{s} \]

which is known as the modified Bessel function. A similar treatment is well-known in physics.

We should note that Eq. (3.22) is useful for determining the Laplace transform when \( a > 0 \). However, application of Eq. (3.22) may lead to an incorrect result. Consider, for example, \( x(t) = \sin(\theta) \), which does not have a unique value \( \theta = \pi \). Application of Eq. (3.22) to Eq. (3.30) leads to a value \( \theta = \pi \), which is incorrect.

**Example 3-6: Applying the Frequency Differentiation Property**

Given that

\[ x(t) = \mathcal{L}^{-1} \{e^{-at} x(t) \} = \frac{1}{8 + a} \]

apply Eq. (3.35) to obtain the Laplace transform of \( e^{-at} x(t) \).

**Solution:**

\[ \mathcal{L} \{ e^{-at} x(t) \} = \frac{d}{ds} \left( \frac{1}{s + a} \right) = \frac{1}{(s + a)^2} \]

**Table 3-1: Properties of the Laplace transform for several functions where \( s > \alpha \)***

<table>
<thead>
<tr>
<th>Property</th>
<th>( x(t) )</th>
<th>( \mathcal{L} { x(t) } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Multiplication by constant</td>
<td>( K \cdot x(t) )</td>
<td>( K \cdot \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>2. Linearity</td>
<td>( x_1(t) + x_2(t) )</td>
<td>( \mathcal{L} { x_1(t) } + \mathcal{L} { x_2(t) } )</td>
</tr>
<tr>
<td>3. Time scaling</td>
<td>( x(at) )</td>
<td>( \frac{1}{a} \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>4. Time shift</td>
<td>( x(t - T) )</td>
<td>( e^{-sT} \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>5. Frequency shift</td>
<td>( e^{\alpha t} x(t) )</td>
<td>( \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>6. Time 1st derivative</td>
<td>( x'(t) )</td>
<td>( s \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>7. Time 2nd derivative</td>
<td>( x''(t) )</td>
<td>( s^2 \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>8. Time integral</td>
<td>( \int_0^t x(t) , dt )</td>
<td>( \frac{1}{s} \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>9. Frequency integral</td>
<td>( \int_0^\infty x(t) , dt )</td>
<td>( \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>10. Initial value</td>
<td>( x(0^+) )</td>
<td>( \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>11. Final value</td>
<td>( \lim_{t \to \infty} x(t) )</td>
<td>( \mathcal{L} { x(t) } )</td>
</tr>
<tr>
<td>12. Convolution</td>
<td>( x_1(t) * x_2(t) )</td>
<td>( \mathcal{L} { x_1(t) } \cdot \mathcal{L} { x_2(t) } )</td>
</tr>
</tbody>
</table>

**Exercise 3-7:** Obtain the Laplace transform of

(a) \( x_1(t) = \frac{2}{(2 - e^{-t})} u(t) \)

(b) \( x_2(t) = e^{-30} \cos(2t + 30) u(t) \)

**Answer:**

(a) \( X_1(s) = \frac{2s + 4}{s^2 + 4} \)

(b) \( X_2(s) = \frac{0.8606s + 1.6}{s^2 + 6s + 12} \)
\[ e^{\theta} \cos \left( \frac{2\pi + \theta}{3} \right) = e^{\theta} \cos \left( \frac{\theta}{3} - \frac{\pi}{3} \right) - \sin \frac{\theta}{3} \sin \frac{\pi}{3} \]

\[ = e^{\theta} \cos \theta - \frac{\sqrt{3}}{2} e^{\theta} \sin \theta \]

\[ e^{2\theta} \cos \left( \frac{\theta}{3} \right) + \frac{2e^{\theta}}{3} \]

\[ e^{\theta} \cos \left( \frac{2\pi + \theta}{3} \right) = \frac{1}{2} \left( \cos \frac{\theta}{3} + \cos \frac{2\pi + \theta}{3} \right) \]

\[ \cos \theta = \frac{1}{2} \left( \cos \frac{\theta}{3} + \cos \frac{2\pi + \theta}{3} \right) \]

\[ \sin \theta = \frac{\sqrt{3}}{2} \left( \sin \frac{\theta}{3} + \sin \frac{2\pi + \theta}{3} \right) \]

\[ \gamma(\theta) = \frac{c}{\theta} \frac{\theta}{\sin(\theta) + \cos(\theta)} \]

\[ \gamma(\theta) = \frac{\frac{1}{\theta}}{\frac{1}{\sin(\theta) + \cos(\theta)}} \]

\[ \gamma(\theta) = \frac{\sin(\theta) + \cos(\theta)}{\theta} \]

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\[
R \cdot q(t) + \left( \int_0^t \frac{1}{c} [v'(t') + v(t')] \, dt' \right) + \frac{\partial V_0}{\partial t} = V_c \cdot v(t)
\]  
\[
(3.46)
\]

**Step 2:** Define the Laplace transform corresponding to the unknown current \(i(t)\), obtain s-domain equivalents for each term in Eq. (3.46), and then apply the linearity property of the Laplace transform to transform the electric differential equation into the s-domain.

The four terms of Eq. (3.46) have the following Laplace transforms:

- \(R \cdot q(t)\) transforms to \(R \cdot q(s)\)
- \(\int_0^t \frac{1}{c} [v'(t') + v(t')] \, dt'\) transforms to \(\frac{1}{s} [V_c \cdot v(s)] + \frac{1}{c} \cdot \frac{V_c}{s}\)
- \(L \cdot \frac{\partial V_0}{\partial t}\) transforms to \(L \cdot \frac{V_0}{s}\)

and

- \(V_c \cdot v(t)\) transforms to \(V_c \cdot v(s)\)

\[
\text{where we have set } V_0(0) = 0.\]

The beauty of the Laplace transform is that it converts an electric differential equation to the time domain into a simple algebraic equation in the s-domain.

**Solving for \(v(t)\):**

For \(E(t)\) in Eq. (3.47) with their respective values faults:

- \(E(t) = \frac{1}{c} [v'(t') + v(t')]\)
- \(\frac{\partial V_0}{\partial t}\) transforms to \(L \cdot \frac{V_0}{s}\)

According to entry 16 in Table 3.2, we have

\[
\mathcal{L}^{-1} \left[ \frac{1}{s} \right] = e^{-at} v(t)
\]

Hence,

\[
(3.47)
\]

**Example:**

\[
V_c(t) = K \cdot \frac{\partial E(t)}{\partial t} + v_c(t) + L \cdot \frac{\partial^2 E(t)}{\partial t^2}
\]

\[
= K \cdot \frac{\partial E(t)}{\partial t} + v_c(t) + \frac{L}{s} \cdot \frac{\partial E(t)}{\partial t}
\]

\[
= \frac{\partial E(t)}{\partial t} \left( K + \frac{L}{s} \right) + v_c(t)
\]

**Figure 3-3:** RLC circuit. The dc source, in combination with the switch, constitutes an input excitation \(V_c(t) = V_0 \cdot v(t)\).

\[
V_c(t) = S \left( C \cdot \frac{\partial E(t)}{\partial t} + \frac{1}{C} \cdot \frac{1}{s} \right)
\]

\[
= S \left( \frac{1}{C} \cdot \frac{1}{s} \right)
\]

**Figure 3-3:** RLC circuit. The dc source, in combination with the switch, constitutes an input excitation \(V_c(t) = V_0 \cdot v(t)\).
For \( H(s) = \frac{0.4}{s^2 + 5s + 12} \)

Let's find its partial fraction expansion.

\[
H(s) = \frac{A}{s-3} + \frac{B}{s+4}
\]

Let \( A = H(s) \cdot s \) and \( B = H(s) \cdot 5 \)

\[
A = \frac{0.4}{(s-3)(s+4)} \bigg|_{s=3} = \frac{0.4}{0} \quad \text{and} \quad B = \frac{0.4}{(s-3)(s+4)} \bigg|_{s=-4} = \frac{0.4}{20} = \frac{1}{50}
\]

\[
\frac{A}{s-3} = H(s) - \frac{1}{5} \quad \frac{B}{s+4} = \frac{0.4}{s(s+5)}
\]

Finally, \( H(s) = \frac{0.4}{s(s+5)} + \frac{0.25}{s+4} \)

\[
\begin{align*}
\frac{d}{dt} e^{at} u(t) & \leftrightarrow \frac{-a}{s+a} \\
\int e^{at} u(t) dt & \leftrightarrow \frac{1}{s+a} \\
\int \int e^{at} u(t) dt & \leftrightarrow \frac{-1}{(s+a)^2}
\end{align*}
\]