ECE 103 Lecture 21, November 21, 2018

Exercise 3-2: Determine the Laplace transform of the causal sawtooth waveform shown in Fig. E3-2 (compare with Example 1-4).

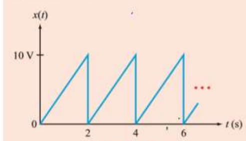

Figure E3-2

Answer:

$$\mathbf{X}(\mathbf{s}) = \mathbf{X}_1(\mathbf{s}) \sum_{n=0}^{\infty} e^{-2n\mathbf{s}} = \frac{\mathbf{X}_1(\mathbf{s})}{1 - e^{-2\mathbf{s}}},$$

where

$$\mathbf{X}_1(\mathbf{s}) = \int_0^2 (5t) e^{-\mathbf{s}t}\, dt = \frac{5}{\mathbf{s}^2}[1 - (2\mathbf{s}+1)e^{-2\mathbf{s}}].$$

3-3.4 Time Differentiation

Differentiating $x(t)$ in the time domain is equivalent to (a) multiplying $\mathbf{X}(\mathbf{s})$ by $\mathbf{s}$ in the s-domain and then (b) subtracting $x(0^-)$ from $\mathbf{s}\,\mathbf{X}(\mathbf{s})$:

$$x' = \frac{dx}{dt} \longleftrightarrow \mathbf{s}\,\mathbf{X}(\mathbf{s}) - x(0^-). \qquad (3.21)$$

(time-differentiation property)

To verify Eq. (3.21), we start with the standard definition for the Laplace transform:

$$\mathcal{L}[x'] = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-\mathbf{s}t}\, dt. \qquad (3.22)$$

$$= x(t)e^{-st}\Big|_{0_-}^{\infty} - \int_{0_-}^{\infty} x(t)\, d\left(e^{-st}\right)$$

$$= 0 - x(0_-) - (-s)\underbrace{\int_{0_-}^{\infty} x(t) e^{-st} dt}_{= X(s)}$$

$$= -x(0_-) + s X(s)$$

Integration by parts with

$$u = e^{-\mathbf{s}t}, \qquad du = -\mathbf{s}e^{-\mathbf{s}t}\, dt,$$
$$dv = \left(\frac{dx}{dt}\right) dt, \qquad \text{and} \qquad v = x$$

gives

$$\mathcal{L}[x'] = uv\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\, du$$
$$= e^{-\mathbf{s}t}\, x(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} -\mathbf{s}\, x(t)\, e^{-\mathbf{s}t}\, dt$$
$$= -x(0^-) + \mathbf{s}\,\mathbf{X}(\mathbf{s}), \qquad (3.23)$$

which is equivalent to Eq. (3.21).

Higher derivatives can be obtained by repeating the application of Eq. (3.21). For the second derivative of $x(t)$, we have

$$x'' = \frac{d^2x}{dt^2} \longleftrightarrow \mathbf{s}^2\,\mathbf{X}(\mathbf{s}) - \mathbf{s}\,x(0^-) - x'(0^-), \qquad (3.24)$$

(second-derivative property)

3-3.5 Time Integration

Integration of $x(t)$ in the time domain is equivalent to dividing $\mathbf{X}(\mathbf{s})$ by $\mathbf{s}$ in the s-domain:

$$\int_0^t x(t')\, dt' \longleftrightarrow \frac{1}{\mathbf{s}}\mathbf{X}(\mathbf{s}). \qquad (3.25)$$

(time-integration property)

Application of the Laplace transform definition gives

$$\mathcal{L}\left[\int_{0^-}^t x(t')\, dt'\right] = \int_{0^-}^{\infty} \underbrace{\left[\int_{0^-}^t x(t')\, dt'\right]}_{u} \underbrace{e^{-\mathbf{s}t}\, dt}_{dv}. \qquad (3.26)$$

Integration by parts with

$$u = \int_{0^-}^t x(t')\, dt', \qquad du = x(t)\, dt,$$
$$dv = e^{-\mathbf{s}t}\, dt, \qquad \text{and} \qquad v = -\frac{e^{-\mathbf{s}t}}{\mathbf{s}}$$

leads to

$$\mathcal{L}\left[\int_{0^-}^t x(t')\, dt'\right]$$
$$= uv\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\, du$$
$$= \underbrace{\left[-\frac{e^{-\mathbf{s}t}}{\mathbf{s}} \int_{0^-}^t x(t')\, dt'\right]\Bigg|_{0^-}^{\infty}}_{= 0 - 0 = 0} + \frac{1}{\mathbf{s}} \int_{0^-}^{\infty} x(t)\, e^{-\mathbf{s}t}\, dt = \frac{1}{\mathbf{s}}\mathbf{X}(\mathbf{s}). \qquad (3.27)$$

Both limits on the first term on the right-hand side yield zero values.

For example, since

$$\delta(t) \longleftrightarrow 1,$$

it follows that

$$u(t) = \int_{0^-}^t \delta(t')\, dt' \longleftrightarrow \frac{1}{\mathbf{s}}$$

and

$$r(t) = \int_{0^-}^t u(t')\, dt' \longleftrightarrow \frac{1}{\mathbf{s}^2}.$$

Application

$$\cos\omega_0 t \xrightarrow{\mathcal{L}} \frac{s}{s^2+\omega_0^2}$$

$$\frac{d}{dt}(\cos\omega_0 t) = -\omega_0 \sin\omega_0 t$$

$$\sin\omega_0 t = -\frac{1}{\omega_0}\frac{d}{dt}(\cos\omega_0 t)$$

$$\downarrow\mathcal{L} \qquad \downarrow\mathcal{L}$$

$$= -\frac{1}{\omega_0}\left[s\left(\frac{s}{s^2+\omega_0^2}\right) - \underbrace{\cos\omega_0(0_-)}_{=1}\right]$$

$$= -\frac{1}{\omega_0}\left[\frac{s^2}{s^2+\omega_0^2} - 1\right]$$

$$= -\frac{1}{\omega_0}\,\frac{\cancel{s^2} - (\cancel{s^2}+\omega_0^2)}{s^2+\omega_0^2} = \frac{\omega_0}{s^2+\omega_0^2}$$

$$= \mathcal{L}[\sin\omega_0 t]$$

3-3.6 Initial- and Final-Value Theorems

The relationship between $x(t)$ and $\mathbf{X}(\mathbf{s})$ is such that the initial value $x(0^+)$ and the final value $x(\infty)$ of $x(t)$ can be determined directly from the expression of $\mathbf{X}(\mathbf{s})$—provided certain conditions are satisfied (as discussed later in this subsection).

Consider the derivative property represented by Eq. (3.23) as

$$\mathcal{L}[x'] = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-\mathbf{s}t}\, dt = \mathbf{s}\,\mathbf{X}(\mathbf{s}) - x(0^-). \qquad (3.28)$$

If we take the limit as $\mathbf{s} \to \infty$ while recognizing that $x(0^-)$ is independent of $\mathbf{s}$, we get

$$\lim_{\mathbf{s}\to\infty}\left[\int_{0^-}^{\infty} \frac{dx}{dt} e^{-\mathbf{s}t}\, dt\right] = \lim_{\mathbf{s}\to\infty}[\mathbf{s}\,\mathbf{X}(\mathbf{s})] - x(0^-). \qquad (3.29)$$

The integral on the left-hand side can be split into two integrals: one over the time segment $(0^-, 0^+)$, for which $e^{-st} = 1$, and another over the segment $(0^+, \infty)$. Thus,

$$\lim_{s\to\infty}\left[\int_{0^-}^{\infty} \frac{dx}{dt}\, e^{-st}\, dt\right] = \lim_{s\to\infty}\left[sX(s) - x(0-)\right]$$

$$= \lim_{s\to\infty}\left[\int_{0^-}^{0^+} \frac{dx}{dt}\, dt + \int_{0^+}^{\infty} \frac{dx}{dt}\, e^{-st}\, dt\right] = x(0^+) - x(0^-). \quad (3.30)$$

$$= \lim_{s\to\infty} sX(s) - \cancel{x(0-)}$$

$$= x(0+) - \cancel{x(0-)}$$

As $s \to \infty$, the exponential function e^{-st} causes the integrand of the last term to vanish. Equating Eqs. (3.29) and (3.30) leads to

$$x(0^+) = \lim_{s\to\infty} s\,\mathbf{X}(s), \quad (3.31)$$

(initial-value theorem)

which is known as the ***initial-value theorem***.

A similar treatment in which s is made to approach 0 (instead of ∞) in Eq. (3.29) leads to the ***final-value theorem***:

$$x(\infty) = \lim_{s\to 0} s\,\mathbf{X}(s). \quad (3.32)$$

(final-value theorem)

We should note that Eq. (3.32) is useful for determining $x(\infty)$, *so long as $x(\infty)$ exists*. Otherwise, application of Eq. (3.32) may lead to an erroneous result. Consider, for example, $x(t) = \cos(\omega_0 t)\, u(t)$, which does not have a unique value as $t \to \infty$. Yet, application of Eq. (3.32) to Eq. (3.9) leads to $x(\infty) = 0$, which is incorrect.

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e.g.) $t\,x(t) = t \xrightarrow{\mathcal{L}} \frac{1}{s^2}$

$x(t) = u(t)$

$X(s) = \frac{1}{s}$

$-\frac{d}{ds}X(s) = -\frac{d}{ds}\left(\frac{1}{s}\right) = -\left(-\frac{1}{s^2}\right) = \frac{1}{s^2}$ ✓

e.g.) $t\,r(t) = t^2 \xrightarrow{\mathcal{L}}$

$R(s) = \mathcal{L}[r(t)] = \frac{1}{s^2}$, $-\frac{d}{ds}R(s) = -\left(-\frac{2s}{(s^2)^2}\right) = \frac{2}{s^3}$

Let us do direct $\mathcal{L}$:

$\mathcal{L}[t^2] = \int_0^\infty t^2 e^{-st} dt = \int_0^\infty t^2 d\left(\frac{e^{-st}}{-s}\right) = t^2\left(\frac{e^{-st}}{-s}\right)\Big|_0^\infty - \int\left(-\frac{e^{-st}}{s}\right)2t\,dt$
$= \frac{2}{s}\mathcal{L}[t] = \frac{2}{s}\frac{1}{s^2} = \frac{2}{s^3}$ ✓

Example 3-5: Applying the Frequency Differentiation Property

Given that

$$\mathbf{X}(s) = \mathcal{L}[e^{-at}\, u(t)] = \frac{1}{s+a},$$

apply Eq. (3.35) to obtain the Laplace transform of $te^{-at}\, u(t)$.

Solution:

$$\mathcal{L}[te^{-at}\, u(t)] = -\frac{d}{ds}\,\mathbf{X}(s) = -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}.$$

3-3.8 Frequency Integration

Integrating both sides of Eq. (3.33) from s to ∞ gives

$$\int_s^\infty \mathbf{X}(s')\, ds' = \int_s^\infty \left[\int_{0^-}^\infty x(t)\, e^{-s't}\, dt\right] ds'. \quad (3.36)$$

Since t and s' are independent variables, we can interchange the order of the integration on the right-hand side of Eq. (3.36),

$$\int_s^\infty \mathbf{X}(s')\, ds' = \int_{0^-}^\infty \left[\int_s^\infty x(t)\, e^{-s't}\, ds'\right] dt$$

$$= \int_{0^-}^\infty \left[\frac{x(t)}{-t}\, e^{-s't}\Big|_s^\infty\right] dt$$

$$= \int_{0^-}^\infty \left[\frac{x(t)}{t}\right] e^{-st}\, dt = \mathcal{L}\left[\frac{x(t)}{t}\right]. \quad (3.37)$$

This ***frequency integration property*** can be expressed as

$$\frac{x(t)}{t} \leftrightarrow \int_s^\infty \mathbf{X}(s')\, ds'. \quad (3.38)$$

(frequency integration property)

Table 3-1: Properties of the Laplace transform for causal functions; i.e., $x(t) = 0$ for $t < 0$.

Property	$x(t)$		$\mathbf{X}(s) = \mathcal{L}[x(t)]$
1. Multiplication by constant	$K\, x(t)$	$\leftrightarrow$	$K\, \mathbf{X}(s)$
2. Linearity	$K_1\, x_1(t) + K_2\, x_2(t)$	$\leftrightarrow$	$K_1\, \mathbf{X}_1(s) + K_2\, \mathbf{X}_2(s)$
3. Time scaling	$x(at), \quad a > 0$	$\leftrightarrow$	$\frac{1}{a}\mathbf{X}\left(\frac{s}{a}\right)$
4. Time shift	$x(t-T)\, u(t-T)$	$\leftrightarrow$	$e^{-Ts}\, \mathbf{X}(s)$
5. Frequency shift	$e^{-at}\, x(t)$	$\leftrightarrow$	$\mathbf{X}(s+a)$
6. Time 1st derivative	$x' = \frac{dx}{dt}$	$\leftrightarrow$	$s\,\mathbf{X}(s) - x(0^-)$
7. Time 2nd derivative	$x'' = \frac{d^2x}{dt^2}$	$\leftrightarrow$	$s^2\mathbf{X}(s) - sx(0^-) - x'(0^-)$
8. Time integral	$\int_{0^-}^t x(t')\, dt'$	$\leftrightarrow$	$\frac{1}{s}\mathbf{X}(s)$
9. Frequency derivative	$t\, x(t)$	$\leftrightarrow$	$-\frac{d}{ds}\mathbf{X}(s) = -\mathbf{X}'(s)$
10. Frequency integral	$\frac{x(t)}{t}$	$\leftrightarrow$	$\int_s^\infty \mathbf{X}(s')\, ds'$
11. Initial value	$x(0^+)$	=	$\lim_{s\to\infty} s\,\mathbf{X}(s)$
12. Final value	$\lim_{t\to\infty} x(t) = x(\infty)$	=	$\lim_{s\to 0} s\,\mathbf{X}(s)$
13. Convolution	$x_1(t) * x_2(t)$	$\leftrightarrow$	$\mathbf{X}_1(s)\, \mathbf{X}_2(s)$

Exercise 3-7: Obtain the Laplace transform of
(a) $x_1(t) = 2(2 - e^{-t})\, u(t)$ and
(b) $x_2(t) = e^{-3t}\cos(2t + 30°)\, u(t)$.

Answer: (a) $\mathbf{X}_1(s) = \dfrac{2s+4}{s(s+1)}$,
(b) $\mathbf{X}_2(s) = \dfrac{0.866s + 1.6}{s^2 + 6s + 13}$. (See S^2)

$$\cos(2t+30^\circ) = \cos 2t \underbrace{\cos 30^\circ}_{=0.866} - \sin 2t \underbrace{\sin 30^\circ}_{=0.5}$$

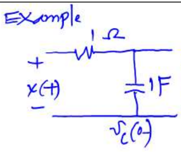
$$e^{-3t}\cos 3t \xrightarrow{\mathcal{L}} \frac{(s+3)}{(s+3)^2+3^2}$$

$$e^{-3t}(\cos 3t \; 0.866 - \sin 3t \; (0.5))$$

$$= \frac{0.866\,(s+3)}{(s+3)^2+3^2} - \frac{3 \times 0.5}{(s+3)^2+3^2}$$

$$= \frac{0.866 s + 3(0.866) - 1.5}{s^2+6s+18}$$

Example



$$x(t) = 1\left(1\,\frac{dy(t)}{dt}\right) + y(t)$$

$$\downarrow \mathcal{L}$$

$$X(s) = \left(sY(s) - y(0_-)\right) + Y(s)$$

$$Y(s) = \frac{X(s) + v_c(0_-)}{s+1}$$

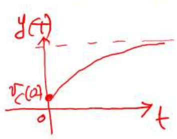
$$y(0_-) = v_c(0_-)$$

For $x(t) = u(t)$ $\quad X(s) = \left(\frac{1}{s}\right)$, $\quad Y(s) = \frac{\left(\frac{1}{s}\right) + v_c(0_-)}{s+1}$

$$= \underbrace{\frac{1}{s(s+1)}}_{= \frac{1}{s} - \frac{1}{s+1}} + v_c(0_-)\frac{1}{s+1}$$

$$\downarrow \mathcal{L}^{-1}$$

$$y(t) = (1 - e^{-t})\,u(t) + v_c(0_-)\,e^{-t}\,u(t)$$



For $x(t) = 1 \sin 10t \; u(t)$

$$X(s) = \left(\frac{10}{s^2+10^2}\right)$$

$$Y(s) = \frac{\left(\frac{10}{s^2+10^2}\right) + v_c(0_-)}{s+1}$$

$$= \left(\frac{10}{(s^2+10^2)(s+1)}\right) + \frac{v_c(0_-)}{s+1}$$

where

$$\frac{10}{(s^2+10^2)(s+1)} = \frac{A}{(s+j10)} + \frac{B = A^*}{(s-j10)} + \frac{C}{s+1}$$

$$C = \frac{10}{(s^2+10^2)(s+1)} \times (s+1)\Big|_{s=-1} = \frac{10}{101}$$

$$A = \frac{10}{(s^2+10^2)(s+1)} \times (s+j10)\Big|_{s=-j10} = \frac{10}{(s-j10)(s+1)}\Big|_{s=-j10}$$

$$= \frac{1\cancel{0}}{-j2\cancel{0}(1-j10)} = \frac{+j}{2(1-j10)}$$

$$A^* = \frac{-j}{2(1+j10)}$$

$$\Rightarrow Y(s) = \frac{\frac{+j}{2(1-j10)}}{s+j10} + \frac{\frac{-j}{2(1+j10)}}{s-j10} + \frac{\frac{10}{101}}{s+1} + \frac{v_c(0)}{s+1}$$

$$= \frac{\frac{+j}{2(1-j10)}(s-j10) - \frac{j}{2(1+j10)}(s+j10)}{(s+j10)(s-j10)} + \frac{\frac{10}{101}}{s+1} + \frac{v_c(0)}{s+1}$$

$$= \frac{\frac{j(s-j10)(1+j10) - j(s+j10)(1-j10)}{2(1-j10)(1+j10)}}{s^2+10^2} + \left(\frac{\frac{10}{101}}{s+1}\right) + \frac{v_c(0)}{s+1}$$

$$= \frac{\frac{1}{2(1+10^2)}\,j\left(s+j10s-j10-(-100)-(s-j10s+j10-(-100)\right)}{s^2+10^2} + \left(\frac{\frac{10}{101}}{s+1}\right)$$

$$+ \frac{v_c(0)}{s+1} = \frac{\frac{1}{2(101)}\,j\left(j20s - j20\right)}{(s^2+10^2)} + \frac{\frac{10}{101}}{s+1} + \frac{v_c(0)}{s+1}$$

$$= \frac{\frac{10}{101}(1-s)}{s^2+10^2} + \frac{\frac{10}{101}}{s+1} + \frac{v_c(0)}{s+1}$$

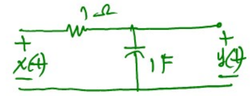
$$\downarrow \mathcal{L}^{-1}$$

$$y(t) = u(t)\left[\underbrace{\frac{1}{101}\sin 10t - \frac{10}{101}\cos 10t}_{} + \left(v_c(0) + \frac{10}{101}\right)e^{-t}\right]$$

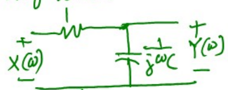
$$= u(t)\left[\frac{1}{\sqrt{101}}\sin\left(10t - \underbrace{\tan^{-1}10}_{=84.29^\circ}\right) + \underbrace{\left(v_c(0_-) + \frac{10}{101}\right)e^{-t}}_{\text{transient}}\right]$$

In steady state (as $t \to \infty$)

$$y_{ss}(t) = \frac{1}{\sqrt{101}}\sin(10t - 84.29^\circ)$$



→ In freq. domain



$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1/j\omega C}{1 + 1/j\omega C} = \frac{1}{j\omega C + 1}$$

For $x(t) = 1 \sin \omega t = 1 \sin 10t$

$\omega = 10$

$$H(\omega = 10) = \frac{1}{1 + j10} = \frac{1}{\sqrt{1+10^2}} e^{j\left(-\tan^{-1}\left(\frac{10}{1}\right)\right)}$$

$$= \frac{1}{\sqrt{101}} e^{j(-84.29^\circ)}$$

$$\Rightarrow y_{ss}(t) = \frac{1}{\sqrt{101}} \sin(10t - 84.29^\circ)$$ ✓ (matches)

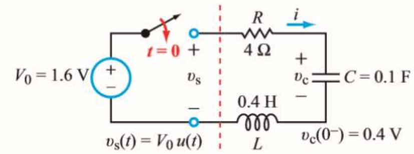


Figure 3-3: RLC circuit. The dc source, in combination with the switch, constitutes an input excitation $v_s(t) = V_0\, u(t)$.

$$R\, i(t) + \left[\frac{1}{C}\int_{0^-}^{t} i(t')\, dt' + v_C(0^-)\right] + L\,\frac{di(t)}{dt} = V_0\, u(t). \tag{3.46}$$

Step 2: Define $\mathbf{I}(s)$ as the Laplace transform corresponding to the unknown current $i(t)$, obtain s-domain equivalents for each term in Eq. (3.46), and then apply the linearity property of the Laplace transform to transform the entire integrodifferential equation into the s-domain.

The four terms of Eq. (3.46) have the following Laplace transform pairs:

$$R\, i(t) \longleftrightarrow R\, \mathbf{I}(s)$$

$$\frac{1}{C}\int_{0^-}^{t} i(t')\, dt' + v_C(0^-) \longleftrightarrow \frac{1}{C}\left[\frac{\mathbf{I}(s)}{s}\right] + \frac{v_C(0^-)}{s}$$

(time integral property)

$$L\,\frac{di(t)}{dt} \longleftrightarrow L[s\,\mathbf{I}(s) - i(0^-)]$$

(time derivative property)

and

$$V_0\, u(t) \longleftrightarrow \frac{V_0}{s}$$

$$R\,\mathbf{I}(s) + \frac{\mathbf{I}(s)}{Cs} + \frac{v_C(0^-)}{s} + Ls\,\mathbf{I}(s) = \frac{V_0}{s}, \tag{3.47}$$

where we have set $i(0^-) = 0$, because no current could have been flowing through the loop prior to closing the switch. Equation (3.47) is the s-domain equivalent of the integrodifferential equation (3.46). Whereas Eq. (3.46) has time-derivatives and integrals, Eq. (3.47) is a simple algebraic equation in s.

► The beauty of the Laplace transform is that it converts an integrodifferential equation in the time domain into a straightforward algebraic equation in the s-domain. ◄

Solving for $\mathbf{I}(s)$ and then replacing R, L, C, V_0, and $v_C(0^-)$ with their numerical values leads to

$$\mathbf{I}(s) = \frac{V_0 - v_C(0^-)}{L\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]} = \frac{1.6 - 0.4}{0.4\left(s^2 + \frac{4}{0.4}s + \frac{1}{0.4 \times 0.1}\right)}$$
$$= \frac{3}{s^2 + 10s + 25} = \frac{3}{(s+5)^2}. \tag{3.48}$$

According to entry #6 in Table 3-2, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^2}\right] = te^{-at}\, u(t).$$

Hence,

$$i(t) = 3te^{-5t}\, u(t). \tag{3.49}$$

Find $v_C(t)$

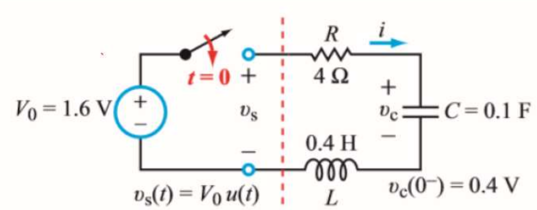


Figure 3-3: RLC circuit. The dc source, in combination with the switch, constitutes an input excitation $v_s(t) = V_0\, u(t)$.

$$v_s(t) = R\left(C\frac{dv_c(t)}{dt}\right) + v_c(t) + L\frac{d}{dt}\left(C\frac{dv_c(t)}{dt}\right)$$

$$v_s(t) = R\left(C\frac{dv_c(t)}{dt}\right) + v_c(t) + L\frac{d}{dt}\left(C\frac{dv_c(t)}{dt}\right)$$

$$= RC\frac{dv_c(t)}{dt} + v_c(t) + LC\frac{d^2v_c(t)}{dt^2}$$

$\downarrow \mathcal{L}$

$$V_s(s) = RC\left(sV_c(s) - v_c(0^-)\right) + V_c(s) + LC\left(s^2V_c(s) - sv_c(0^-) - v_c'(0^-)\right)$$

$$= 0.4\left(sV_c(s) - 0.4\right) + V_c(s) + 0.04\left(s^2V_c(s) - 0.4s - 0\right)$$

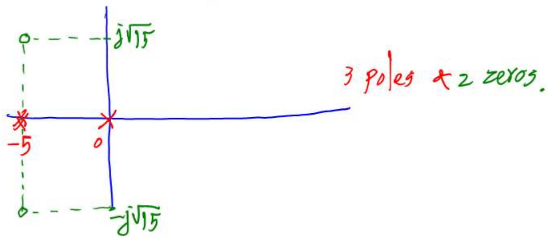
$$= 0.04s^2V_c(s) + 0.4sV_c(s) + V_c(s) - 0.16 - 0.016s = \frac{1.6}{s}$$

$$V_c(s) = \left(\frac{1.6}{s} + 0.016s + 0.16\right)\frac{1}{0.04s^2 + 0.4s + 1}$$

$$= \frac{0.16}{0.04}\,\frac{\frac{1}{s} + 0.1s + 1}{s^2 + 10s + 25} = 4\,\frac{0.1s^2 + s + 1}{s(s+5)^2}$$

$$= 0.4\,\frac{s^2+10s+10}{s(s+5)^2} = 0.4\,\frac{(s+5)^2-15}{s(s+5)^2}$$

$$= 0.4\,\frac{(s+5+j\sqrt{15})(s+5-j\sqrt{15})}{s(s+5)^2}$$



3 poles + 2 zeros.

3-5 Partial Fraction Expansion

Let us assume that, after transforming the integrodifferential equation associated with a system of interest to the s-domain and then solving it for the output signal whose behavior we wish to examine, we end up with an expression $\mathbf{X}(s)$. Our next step is to inverse transform $\mathbf{X}(s)$ to the time domain, thereby completing our solution. The degree of mathematical difficulty associated with the implementation of the inverse transformation depends on the mathematical form of $\mathbf{X}(s)$.

Consider for example the expression

$$\mathbf{X}(s) = \frac{4}{s+2} + \frac{6}{(s+5)^2} + \frac{8}{s^2+4s+5}. \qquad (3.50)$$

The inverse transform, $x(t)$, is given by

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[\mathbf{X}(s)] \\ &= \mathcal{L}^{-1}\left[\frac{4}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{6}{(s+5)^2}\right] \\ &\quad + \mathcal{L}^{-1}\left[\frac{8}{s^2+4s+5}\right]. \end{aligned} \qquad (3.51)$$

By comparison with the entries in Table 3-2, we note the following:

(a) The first term in Eq. (3.51), $4/(s+2)$, is functionally the same as entry #3 in Table 3-2 with $a = 2$. Hence,

$$\mathcal{L}^{-1}\left[\frac{4}{s+2}\right] = 4e^{-2t}\,u(t). \qquad (3.52a)$$

(b) The second term, $6/(s+5)^2$, is functionally the same as entry #6 in Table 3-2 with $a = 5$. Hence,

$$\mathcal{L}^{-1}\left[\frac{6}{(s+5)^2}\right] = 6te^{-5t}\,u(t). \qquad (3.52b)$$

(c) The third term $8/(s^2+4s+5)$, is similar (but not identical) in form to entry #13 in Table 3-2. However, it can be rearranged to assume the proper form:

$$\frac{8}{s^2+4s+5} = \frac{8}{(s+2)^2+1}.$$

Consequently,

$$\mathcal{L}^{-1}\left[\frac{8}{(s+2)^2+1}\right] = 8e^{-2t}\sin t\,u(t). \qquad (3.52c)$$

For $H(s) = \dfrac{0.4(s^2+5s+10)}{s(s+5)^2}$

Let's find its partial fraction expansion

$$H(s) = \frac{A}{s} + \frac{Bs+C}{(s+5)^2}$$

$$A = H(s)\cdot s\Big|_{s=0} = \frac{0.4(s^2+5s+10)}{\cancel{s}(s+5)^2}\cancel{s}\Big|_{s=0} = \frac{0.4(0+0+10)}{(0+5)^2} = \frac{4}{25}$$

$$\frac{As+B}{(s+5)^2} = H(s) - \frac{\frac{4}{25}}{s} = \frac{0.4s^2+2s+4}{s(s+5^2)} - \frac{\frac{4}{25}(s+5)^2}{s(s+5)^2}$$

$$\frac{As+B}{(s+5)^2} = \frac{0.4s^2+2s+4-(0.16s^2+1.6s+4)}{s(s+5)^2}$$

$$= \frac{0.24s^2+3.6s}{s(s+5)^2} = \frac{\cancel{s}(0.24s+3.6)}{\cancel{s}(s+5)^2}$$

$\Rightarrow A = 0.24,\ B = 3.6$

Finally

$$H(s) = \frac{\left(\frac{4}{25}\right)}{s} + \frac{0.24s}{(s+5)^2} + \frac{0.36}{(s+5)^2}$$

$$u(t) \longleftrightarrow \frac{1}{s}$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$$

$$-t\,e^{-at}u(t) \longleftrightarrow \frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{-1}{(s+a)^2}$$

$$+t\,e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^2}$$

$$\underbrace{\frac{d}{dt}\left(t\,e^{-at}u(t)\right)} \longleftrightarrow \frac{s}{(s+a)^2} \qquad \frac{dx}{dt} \longleftrightarrow sX(s) - x(0^-)$$

$$= e^{-at}u(t) - at\,e^{-at}u(t) + t\,e^{-at}\delta(t)$$

$$= \boxed{e^{-at}u(t) - at\,e^{-at}u(t)} + 0$$

$$\downarrow \mathcal{L}$$

$$\frac{1}{s+a} - a\,\frac{1}{(s+a)^2} = \frac{s+a-a}{(s+a)^2} = \frac{s}{(s+a)^2}\ ✓$$