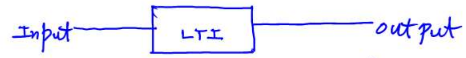
ECE103 Lecture 22, Nov. 21, 2018

Quiz 8 on next Wednesday Nov 28
(instead of Monday)
HW7 & 8 based

Quiz 6 Average = 4.52
σ = 1.84

Input-output Relationship of LTI Systems



$\delta(t) \longrightarrow h(t)$

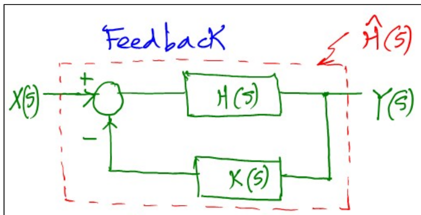
$\mathcal{L}[\delta(t)] = 1 \longrightarrow \mathcal{L}[h(t)] = H(s)$

$x(t) \longrightarrow y(t) = h(t) * x(t) = x(t) * h(t)$

$X(s) \longrightarrow Y(s) = H(s)X(s)$

$X(\omega) \longrightarrow Y(\omega) = H(\omega)X(\omega) = |H(\omega)| e^{j\angle H(\omega)} |X(\omega)| e^{j\angle X(\omega)} = |H(\omega)| |X(\omega)| e^{j(\angle H + \angle X)}$

Feedback



$Y(s) = H(s)\left[X(s) - K(s)Y(s)\right]$

$Y(s)\left(1 + K(s)H(s)\right) = H(s)X(s)$

$\dfrac{Y(s)}{X(s)} = \dfrac{H(s)}{1 + K(s)H(s)} = \hat{H}(s)$

3.10 Determine $x(0^+)$ and $x(\infty)$ given that

$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \dfrac{s^2+4}{2s^2+4s+10} = \dfrac{1}{2}$

$$\mathbf{X(s)} = \frac{\mathbf{s}^2 + 4}{2\mathbf{s}^3 + 4\mathbf{s}^2 + 10\mathbf{s}} .$$

$x(\infty) = \lim_{s\to 0} sX(s) = \dfrac{s(s^2+4)}{2s^3+4s^2+10s}\Big|_{s=0} = \dfrac{0+4}{2\times0+4\times0+10} = \dfrac{4}{10} = 0.4$

***3.11** Determine $x(0^+)$ and $x(\infty)$ given that

$$\mathbf{X(s)} = \frac{12e^{-2\mathbf{s}}}{\mathbf{s}(\mathbf{s}+2)(\mathbf{s}+3)} .$$

$\dfrac{s^2+4}{2s^3+4s^2+10s} = \dfrac{s^2+4}{2s(s^2+4s+5)}$

$= \dfrac{s^2+4}{2s\left((s+2)^2+1\right)} = \dfrac{(s+j2)(s-j2)}{2s(s+2+j1)(s+2-j1)}$

$= \left[\dfrac{A}{s} + \dfrac{B}{s+2+j1} + \dfrac{B^*}{s+2-j1}\right]$

For A, multiply by s and set s = 0

$A = \dfrac{s(s^2+4)}{2s^3+4s^2+10s} = \dfrac{0+4}{0+0+10} = \dfrac{2}{5}$

$\dfrac{s^2+4}{2s^3+4s^2+10s} - \dfrac{\frac{2}{5}}{s}$

$= \dfrac{s^2+4 - \frac{2}{5}(2s^2+4s+10)}{s(2s^2+4s+10)}$

$= \dfrac{s^2+4 - \frac{4}{5}s^2 - \frac{8}{5}s - 4}{s(2s^2+4s+10)} = \dfrac{\frac{1}{5}s^2 - \frac{8}{5}s}{s(2s^2+4s+5)}$

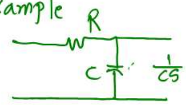
$= \dfrac{\frac{1}{5}(s-8)}{2s^2+4s+10} = \dfrac{1}{10}\left[\dfrac{(s-8) = (s+1) - 9}{(s+1)^2 + 2^2}\right]$, where $2s^2+4s+10 = 2(s^2+2s+5)$

$\dfrac{\omega_0}{s^2+\omega_0^2} \leftrightarrow \sin\omega_0 t$

$\dfrac{-9\left(= -2 \times \frac{9}{2}\right)}{(s+1)^2+2^2}$

$\xrightarrow{\mathcal{L}^{-1}} \dfrac{2}{5}u(t) + \dfrac{1}{10}\left[e^{-t}\cos 2t - \dfrac{9}{2}e^{-t}\sin t\right]u(t)$

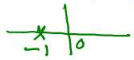
Example

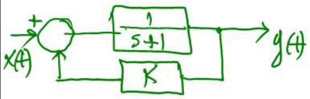


$$H(s) = \frac{\frac{1}{Cs}}{\frac{1}{Cs} + R} = \left(\frac{1}{RC}\right)\frac{1}{s + \frac{1}{RC}}$$

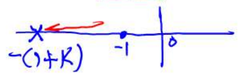
If $R = 1\,M\Omega$, $C = 1\,\mu F$, then $RC = 10^{6}(10^{-6}) = 1$

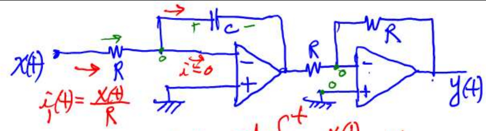
$$H(s) = \frac{1}{s+1}$$





$$\hat{H}(s) = \frac{\left(\frac{1}{s+1}\right)}{1 + K\left(\frac{1}{s+1}\right)} = \frac{1}{s+1+K} = \frac{1}{s+(1+K)}$$



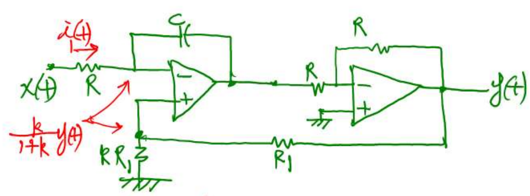


$$y(t) = \frac{1}{C}\int_0^t \frac{x(t)}{R}\,dt$$

$$Y(s) = \frac{1}{RCs}X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{RCs}$$

Non-inverting Integrator



$$i_1(t) = \frac{x(t) - \frac{k}{1+k}y(t)}{R}$$

$$y(t) = \frac{1}{C}\int_{0_-}^{t}\frac{1}{R}\left(x(t) - \frac{k}{1+k}y(t)\right)dt$$

$\downarrow \mathcal{L}$

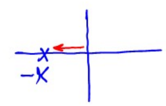
$$Y(s) = \frac{1}{RC}\frac{1}{s}\left[X(s) - K\,Y(s)\right]$$

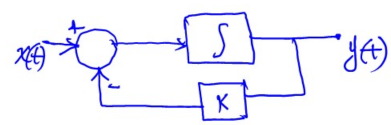
Let $\begin{cases}\frac{k}{1+k} = K \\ RC = 1\end{cases}$

$$\Rightarrow Y(s) = \frac{1}{s}\left[X(s) - K\,Y(s)\right]$$

$$s\,Y(s) + K\,Y(s) = X(s)$$

$$\hat{H}(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+K}$$

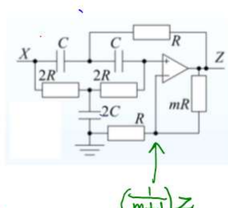




Second-Order Butterworth Filter

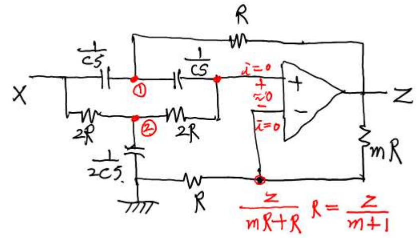
$$H(\omega) = \frac{Z(\omega)}{X(\omega)} = \frac{(1+m)\left((2j\omega RC)^2 + 1\right)}{(2j\omega RC)^2 + 4(1-m)j\omega RC + 1}$$

Here m is the ratio of the two feedback resistance which determines the gain and quality for the filter. The drop frequency of this twin-T notch filter is $f_{drop} = 1/4\pi RC$. For designing a 60 Hz drop filter, let's use R=10 kΩ and C=133 nF.



$$H(s) = \frac{(1+m)\left[(2RCs)^2 + 1\right]}{(2RCs)^2 + 4(1-m)RCs + 1}$$

Derivation of $H(s) = \frac{Z(s)}{X(s)}$ for the Butterworth filter below:



KCL at ①

$$(X - V_1)Cs = \left(V_1 - \frac{Z}{m+1}\right)Cs + (V_1 - Z)\frac{1}{R}$$

Multiplying R on both sides ⇒

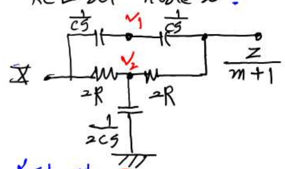
$$(X - V_1)RCs = \left(V_1 - \frac{Z}{m+1}\right)RCs + V_1 - Z$$

$$X RCs = (1 + 2RCs)V_1 - \left(\frac{1}{m+1}RCs + 1\right)Z \quad \text{Eq(I)}$$

If we can express V_1 in terms of X, Z then Eq (I) can be used to find $H(s)$.

Next let us consider node 2 :

KCL at node 2.



KCL at ②

$$\frac{X - V_2}{2R} = V_2(2Cs) + \left(V_2 - \frac{Z}{m+1}\right)\frac{1}{2R}$$

Multiplying 2R on both sides ⇒

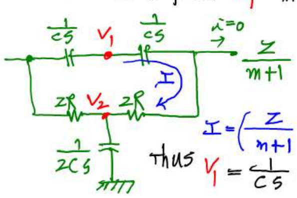
$$X - V_2 = V_2\, 4RCs + V_2 - \frac{Z}{m+1}$$

$$X = (2 + 4RCs)V_2 - \frac{Z}{m+1}$$

or $V_2 = \dfrac{X + \frac{Z}{m+1}}{2 + 4RCs}$ Eq (II)

Note: V_2 is expressed in X, Z only !

How can we express V_1 in terms of X, Z ?



$$I = \left(\frac{Z}{m+1} - V_2\right)\frac{1}{2R}$$

thus $V_1 = \frac{1}{Cs} I + \frac{Z}{m+1}$

So, $V_1 = \frac{1}{Cs}\left(\frac{Z}{m+1} - V_2\right)\frac{1}{2R} + \frac{Z}{m+1}$

$$= \frac{1}{2RCs}\left(\frac{Z}{m+1} - \frac{X + \frac{Z}{m+1}}{2 + 4RCs}\right) + \frac{Z}{m+1} \quad \text{(Eq III)}$$

Recall Eq (I) :

$$X RCs = (1 + 2RCs)V_1 - \left(\frac{RCs}{m+1} + 1\right)Z$$

$$\overset{\text{(Eq. III)}}{=} (1+2RCs)\left(\frac{1}{2RCs}\left(\frac{Z}{m+1} - \frac{X + \frac{Z}{m+1}}{2+4RCs}\right) + \frac{Z}{m+1}\right) - \left(\frac{RCs}{m+1} + 1\right)Z \quad \text{(Eq. IV)}$$

Next we find $\frac{Z(s)}{X(s)} = H(s)$:

From Eq. (IV),

$$X RCs = \frac{Z}{m+1}\cdot\frac{1+2RCs}{2RCs} - \frac{X + \frac{Z}{m+1}}{2(2RCs)} + \frac{Z}{m+1}(1 + 2RCs) - \left(\frac{RCs}{m} + 1\right)Z$$

combine X terms

$$X\left(RCs + \frac{1}{4RCs}\right) = \frac{Z}{m+1}\left[\frac{1+2RCs}{2RCs} - \frac{1}{4RCs} + (1 + 2RCs) - RCs - (m+1)\right]$$

$$\frac{X((2RCs)^2 + 1)}{\cancel{4RCs}} = \frac{Z}{m+1}\frac{1}{\cancel{4RCs}}\left[2(1+2RCs) - 1 + ((1 + RCs) - (m+1))4RCs\right]$$

$$= \frac{Z}{m+1}\left(1 + 4RCs + (RCs - m)4RCs\right)$$

$$= \frac{Z}{m+1}\left[1 + 4RCs(1-m) + (2RCs)^2\right]$$

$$= X\left[(2RCs)^2 + 1\right]$$

thus

$$\boxed{\frac{Z(s)}{X(s)} = \frac{(m+1)\left[(2RCs)^2 + 1\right]}{(2RCs)^2 + 4(1-m)RCs + 1}} = H(s)$$

$$H(\omega) = H(s)\big|_{s=j\omega} = \frac{(m+1)\left[(2j\omega RC)^2 + 1\right]}{\left[(2j\omega RC)^2 + 4(1-m)j\omega RC + 1\right]}$$

$H(\omega) = 0$ when $(2j\omega RC)^2 = -1$ or $\omega = \frac{1}{2RC}$

$(2RCs)^2+1=0 \Rightarrow s=\pm j\frac{1}{2RC}$ (zeros)

denominator term

$(2RCs)^2+4(1-m)RCs+1=0$

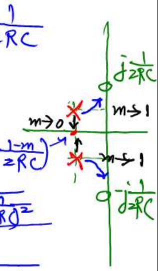
$s^2+\frac{(1-m)}{RC}s+\frac{1}{(2RC)^2}=0$

$s=\frac{-\left(\frac{1-m}{RC}\right)\pm\sqrt{\left(\frac{1-m}{RC}\right)^2-4\frac{1}{(2RC)^2}}}{2}$

$=-\frac{1}{2RC}\left[(1-m)\pm\sqrt{(1-m)^2-1}\right]$

$=-\frac{1}{2RC}\left[(1-m)\pm j\sqrt{1-(1-m)^2}\right]$

Poles



3.43 An LTI system has the LCCDE description

$s^2Y(s)+7sY(s)+12Y(s)=sX(s)+2X(s)$

$$\frac{d^2y}{dt^2}+7\frac{dy}{dt}+12y=\frac{dx}{dt}+2x.$$

$(s^2+7s+12)Y(s)=(s+2)X(s)$

Compute each of the following:

(a) Frequency response function $\mathbf{H}(\omega)$

(b) Poles and zeros of the system

(c) Impulse response $h(t)$

*(**d**) Response to input $x(t)=e^{-2t}\,u(t)$

$H(s)=\frac{Y(s)}{X(s)}=\frac{s+2}{s^2+7s+12}$

$s^2+7s+12=(s+3)(s+4)$

$H(s)=\frac{s+2}{(s+3)(s+4)}=\frac{A}{s+3}+\frac{B}{s+4}$

$A:\ H(s)(s+3)\Big|_{s=-3}=\frac{s+2}{s+4}\Big|_{s=-3}=\frac{-3+2}{-3+4}=-1$

$B:\ H(s)(s+4)\Big|_{s=-4}=\frac{s+2}{s+3}\Big|_{s=-4}=\frac{-4+2}{-4+3}=2$

$H(s)=\frac{-1}{s+3}+\frac{2}{s+4}$

$\downarrow \mathcal{L}^{-1}$

$h(t)=\left(-e^{-3t}+2e^{-4t}\right)u(t)$

$h(t=0^+)=-1+2=+1$

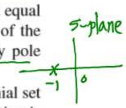
$\lim_{s\to\infty} sH(s)=\lim_{s\to\infty}\frac{s(s+2)}{(s+3)(s+4)}=1$

$h(\infty)=0 \quad \lim_{s\to 0} sH(s)=\lim_{s\to 0}\frac{s(s+2)}{(s+3)(s+4)}=0$ ✓

If The transfer function is ~~then~~

$$\mathbf{H}(s)=\frac{\mathbf{Y}(s)}{\mathbf{X}(s)}=\frac{s+2}{s^2+3s+2}=\frac{s+2}{(s+1)(s+2)}=\frac{1}{s+1}. \tag{3.156}$$

The zeros are the roots of the numerator polynomial set equal to zero, so there are no zeros. The poles are the roots of the denominator polynomial set equal to zero, so the only pole is -1. The system is BIBO stable.



The modes are the roots of the characteristic polynomial set equal to zero. From Eq. (2.123), the characteristic equation is the polynomial whose coefficients are the coefficients of the left side of the LCCDE. In this example, the characteristic equation is

$$s^2+3s+2=0, \tag{3.157}$$

so the modes are $\{-1,-2\}$. Note that modes and poles are *not* the same in this example, due to the cancellation of the factors $(s+2)$ in the numerator and denominator of Eq. (3.156). This is called pole-zero cancellation.

(b) The zero-state response is found by setting all initial conditions to zero and setting the input to $x(t)=4\cos(t)\,u(t)$. The Laplace transform of $x(t)$ is

$$\mathbf{X}(s)=\mathcal{L}[4\cos(t)\,u(t)]=\frac{4s}{s^2+1}, \tag{3.158}$$

and the Laplace transform of the zero-state response is

$$\mathbf{Y}_{\text{ZSR}}(s)=\mathbf{H}(s)\,\mathbf{X}(s)=\left[\frac{1}{s+1}\right]\left[\frac{4s}{s^2+1}\right]. \tag{3.159}$$

Partial fraction expansion of $\mathbf{Y}_{\text{ZSR}}(s)$ gives

$$\mathbf{Y}_{\text{ZSR}}(s)=\frac{1-j}{s-j}+\frac{1+j}{s+j}-\frac{2}{s+1}, \tag{3.160}$$

and the inverse Laplace transform of $\mathbf{Y}_{\text{ZSR}}(s)$ is

$$y_{zsr}(t)=(1-j)e^{jt}\,u(t)+(1+j)e^{-jt}\,u(t)-2e^{-t}\,u(t). \tag{3.161}$$

Noting that $(1\pm j)=\sqrt{2}\,e^{\pm j45^\circ}$, Eq. (3.161) can be put into trigonometric form using entry #3 in Table 3-3. The result is

$$y_{zsr}(t)=2\sqrt{2}\cos(t-45^\circ)\,u(t)-2e^{-t}\,u(t). \tag{3.162}$$

3.44 An LTI system has the LCCDE description

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+13y=\frac{dx}{dt}+2x.$$

Compute each of the following:

(a) Frequency response function $\mathbf{H}(\omega)$

(b) Poles and zeros of the system

(c) Impulse response $h(t)$

(d) Response to input $x(t)=e^{-2t}\,u(t)$