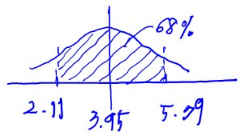
# ECE 103 Lecture 23, Nov. 26 2018

Quiz 8 on this Wednesday Nov. 28

Based on HW 7 & 8

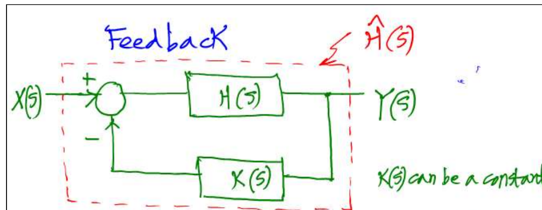
Quiz 6 Average = 4.52, $\sigma = 1.84$

Quiz 7 3.95 1.84




---

## Feedback




$$Y(s) = H(s)\left[X(s) - K(s)Y(s)\right]$$

$$Y(s)\left(1 + K(s)H(s)\right) = H(s)X(s)$$

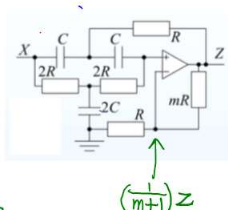
$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + K(s)H(s)} = \hat{H}(s)$$

---

## Second-Order Butterworth Filter

$$H(\omega) = \frac{Z(\omega)}{X(\omega)} = \frac{(1+m)\left((2j\omega RC)^2 + 1\right)}{(2j\omega RC)^2 + 4(1-m)j\omega RC + 1}$$

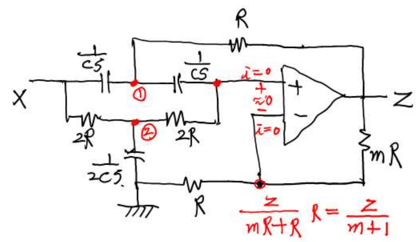
Here m is the ratio of the two feedback resistance which determines the gain and quality for the filter. The drop frequency of this twin-T notch filter is $f_{drop} = 1/4\pi RC$. For designing a 60 Hz drop filter, let's use R=10 kΩ and C=133 nF.




$$H(s) = \frac{(1+m)\left[(2RCs)^2 + 1\right]}{(2RCs)^2 + 4(1-m)RCs + 1}$$

---

Derivation of $H(s) = \frac{Z(s)}{X(s)}$ for the Butterworth filter below:




---

We expressed $V_{②}$ in terms of $X$ and $Z$, then $V_{①}$ in terms of $V_{②}$, thus in terms of $X$, $Z$. By taking the ratio of $Z(s)/X(s)$, we get

$$\boxed{\frac{Z(s)}{X(s)} = \frac{(m+1)\left[(2RCs)^2 + 1\right]}{(2RCs)^2 + 4(1-m)RCs + 1} = H(s)}$$

$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{(m+1)\left[(2j\omega RC)^2 + 1\right]}{\left[(2j\omega RC)^2 + 4(1-m)j\omega RC + 1\right]}$$

---

$$(2RCs)^2 + 1 = 0 \xRightarrow{\text{zeros}} s = \pm j\frac{1}{2RC}$$

denominator

$$(2RCs)^2 + 4(1-m)RCs + 1 = 0$$

$$s^2 + \frac{(1-m)}{RC}s + \frac{1}{(2RC)^2} = 0$$

$$s = \frac{-\left(\frac{1-m}{RC}\right) \pm \sqrt{\left(\frac{1-m}{RC}\right)^2 - 4\frac{1}{(2RC)^2}}}{2}$$

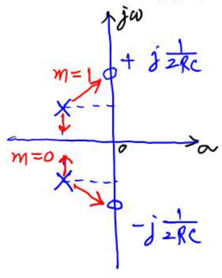
$$= -\frac{1}{2RC}\left[(1-m) \pm \sqrt{(1-m)^2 - 1}\right]$$

$$= -\frac{1}{2RC}\left[(1-m) \pm j\sqrt{1-(1-m)^2}\right]$$

Poles

Zeros at $\pm j\frac{1}{2RC}$

Poles at $-\frac{1}{2RC}\left((1-m) \pm j\sqrt{1-(1-m)^2}\right)$




$H(s) = \frac{(1+m)\left[(2RCs)^2+1\right]}{(2RCs)^2+4(1-m)RCs+1}$

- $\boxed{m=0}$ $H(s) = \frac{(2RCs)^2+1}{(2RCs)^2+4RCs+1}$ $= (2RCs+1)^2$

2 Poles at $s = -\frac{1}{2RC}$

- $\boxed{m=1}$ $H(s) = \frac{(1+1)\left[(2RCs)^2+1\right]}{(2RCs)^2+0\,RCs+1} = 2$

Star tracking Telescope

2 motors for elevation & azimuth

Azimuth (Rotation case)

$H(s) = \frac{b}{s(s+a)}$

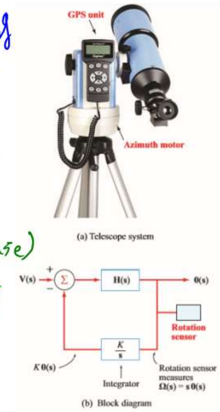
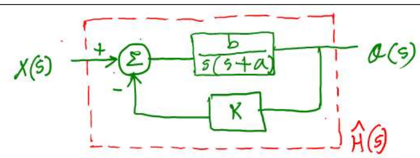



**Figure 4-36:** Telescope (Courtesy of Telescopes.com).




### 4-11.3 Closed-Loop Configuration

The telescope automatic pointing system shown in **Fig. 4-36(a)** uses two motors: one to rotate the telescope in elevation and another to rotate it in azimuth. Its global positioning system (GPS) determines the telescope's coordinates, and its computer contains celestial coordinates for a large number of stars and

galaxies. When a particular star is selected, the computer calculates the angular rotations that the shafts of the two motors should undergo in order to move the telescope from its initial direction to the direction of the selected star. Our goal is to demonstrate how feedback can be used to realize the required rotation automatically. We will limit our discussion to only one of the motors.

$\theta$

The feedback loop is shown in **Fig. 4-36(b)**. A magnetic sensor positioned close to the motor's shaft measures the angular velocity $\omega$. Since $\omega = d\theta/dt$ and differentiation in the time domain is equivalent to multiplication by $\mathbf{s}$ in the $\mathbf{s}$-domain, the $\mathbf{s}$-domain quantity measured by the sensor is

$\Omega$

$$\mathbf{\Omega}(\mathbf{s}) = \mathbf{s}\,\mathbf{\theta}(\mathbf{s}). \qquad (4.146)$$

To perform the desired feedback, we need to feed $K\,\mathbf{\theta}(\mathbf{s})$ into the negative terminal of the summer. Hence, an integrator circuit with transfer function $K/\mathbf{s}$ is used in the feedback arm of the loop. The net result of the process is equivalent to using a feedback loop with $\mathbf{G}(\mathbf{s}) = K$. Consequently, the closed-loop transfer function is

$H(s) = \frac{b}{s^2+as}$

$$\hat{H}(s)\ \cancel{\mathbf{Q}(\mathbf{s})} = \frac{\mathbf{H}(\mathbf{s})}{1+K\,\mathbf{H}(\mathbf{s})}. \qquad (4.147)$$

Use of Eq. (4.139) for $\mathbf{H}(\mathbf{s})$ leads to

$$\hat{H}(s)\ \cancel{\mathbf{Q}(\mathbf{s})} = \frac{b/(\mathbf{s}^2+a\mathbf{s})}{1+Kb/(\mathbf{s}^2+a\mathbf{s})} = \frac{b}{\mathbf{s}^2+a\mathbf{s}+bK}. \qquad (4.148)$$

For $V(s) = \frac{1}{s}$

$\theta(s) = \frac{1}{s}\hat{H}(s)$

$= \frac{b}{s(s^2+as+bK)}$

$= \frac{A}{s} + \frac{B}{s-p_1} + \frac{C}{s-p_2}$

To find $\theta(t)$ in steady state, find $\theta(\infty)$.

$$\theta(t) = \left[\frac{b}{p_1p_2} + \frac{b}{p_1(p_1-p_2)}e^{p_1t} + \frac{b}{p_2(p_2-p_1)}e^{p_2t}\right]u(t). \qquad (4.153)$$

If $p_1$ and $p_2$ are both in the open left half-plane, the second and third term will decay to zero as a function of time, leaving behind

$$\lim_{t\to\infty}\theta(t) = \frac{b}{p_1p_2}. \qquad (4.154)$$

Use of the expressions for $p_1$ and $p_2$ given by Eq. (4.150) in Eq. (4.154) leads to

$$\lim_{t\to\infty}\theta(t) = \frac{1}{K}. \qquad (4.155)$$

$\theta(\infty) = \lim_{s\to 0} s\theta(s)$

$= \lim_{s\to 0} \hat{H}(s)$

$= \left.\frac{b}{s^2+as+bK}\right|_{s=0}$

$= \frac{1}{K}$

If the magnitude of $K$ is on the order of inverse seconds, the final value of $\theta(t)$ is approached very quickly. This final value is equal to the total angular rotation required to move the telescope from its original direction to the designated new direction.

Note that Eq. (4.154) can also be obtained directly from Eq. (4.148) using the final value theorem of the Laplace transform. Application of property #12 in **Table 3-1** gives

$$\lim_{t\to\infty}\theta(t) = \lim_{\mathbf{s}\to 0}\mathbf{s}\,\mathbf{\theta}(\mathbf{s}) = \mathbf{s}\left.\frac{b}{\mathbf{s}(\mathbf{s}^2+a\mathbf{s}+bK)}\right|_{\mathbf{s}=0} = \frac{1}{K} \qquad (4.156)$$

*provided the system is BIBO stable.* The open-loop system given by Eq. (4.139) is BIBO-unstable because one of the poles is at the origin. But the closed-loop system is stable if $K > 0$. Moreover, the closed-loop steady-state unit step response is $\theta(\infty) = 1/K$, independent of $a$ and $b$.

Problem

**4.3** Determine $v_{\text{out}}(t)$ in the circuit in **Fig. P4.3** given that $v_s(t) = 35u(t)$ V, $v_{C_1}(0^-) = 20$ V, $R_1 = 1\ \Omega$, $C_1 = 1$ F, $R_2 = 0.5\ \Omega$, and $C_2 = 2$ F.

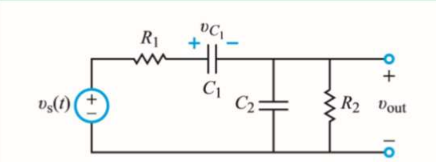
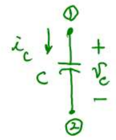



**Figure P4.3:** Circuit for Problem 4.3.



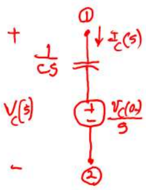

$$i_c(t) = C\frac{dv_c(t)}{dt} \rightarrow I_c(s) = C[sV_c(s) - v_c(0-)]$$

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(\tau)\,d\tau$$

$$= \frac{1}{C}\int_{-\infty}^{0-} i_c(\tau)\,d\tau + \frac{1}{C}\int_{0-}^{t} i_c(\tau)\,d\tau$$

$$= v_c(0-) + \frac{1}{C}\int_{0-}^{t} i_c(\tau)\,d\tau$$

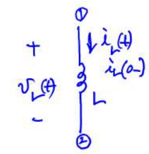
$$\downarrow \mathcal{L}$$




$$V_c(s) = \frac{v_c(0-)}{s} + \left[\frac{1}{Cs}\right] I_c(s)$$

$$\left[V_c(s) - \frac{v_c(0-)}{s}\right] = \frac{1}{Cs} I_c(s)$$

$$I_c(s) = C[sV_c(s) - v_c(0-)]$$



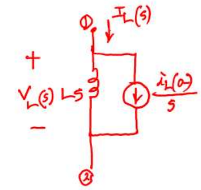

$$v_L(t) = L\frac{di_L(t)}{dt} \rightarrow V_L(s) = L[sI_L(s) - i_L(0-)]$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)\,d\tau$$

$$= \frac{1}{L}\int_{-\infty}^{0-} v_L(\tau)\,d\tau + \frac{1}{L}\int_{0-}^{t} v_L(\tau)\,d\tau$$

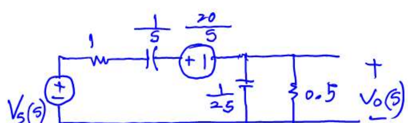
$$= i_L(0-) + \frac{1}{L}\int_{0-}^{t} v_L(\tau)\,d\tau$$

$$\downarrow \mathcal{L}$$




$$I_L(s) = \frac{i_L(0-)}{s} + \frac{1}{Ls} V_L(s)$$

$$V_L(s) = L[sI_L(s) - i_L(0-)]$$




$$= \frac{35}{s}$$

$$V_o(s) = \left(\frac{1}{2+2s}\right)\left[\frac{V_s(s) - \frac{20}{s} - V_o(s)}{1 + \frac{1}{s}}\right]$$

$$= \frac{s}{2(s+1)(s+1)}\left[\frac{35}{s} - \frac{20}{s} - V_o(s)\right]$$

$$= \frac{s}{2(s+1)^2}\left[\frac{15}{s} - V_o(s)\right]$$

$$V_o(s)\left[1 + \frac{s}{2(s+1)^2}\right] = \frac{\cancel{s}}{2(s+1)^2}\frac{15}{\cancel{s}}$$

$$V_o(s)\left[\frac{2(s+1)^2 + s}{\cancel{2(s+1)^2}}\right] = \frac{15}{\cancel{2(s+1)^2}}$$

$$V_o(s) = \frac{15}{2s^2+5s+2} = \frac{15}{2}\,\frac{1}{s^2 + \frac{5}{2}s + 1}$$

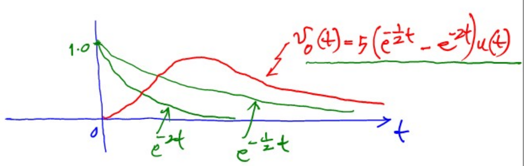
$$= \frac{15}{2}\,\frac{1}{(s+\frac{5}{4})^2 - \frac{9}{16}}$$

$$= \frac{15}{2}\,\frac{1}{(s+\frac{5}{4}+\frac{3}{4})(s+\frac{5}{4}-\frac{3}{4})}$$

$$= \frac{15}{2}\,\frac{1}{(s+2)(s+\frac{1}{2})} = \frac{15}{2}\left(\frac{A}{s+2} + \frac{B}{s+\frac{1}{2}}\right)$$

$$A = \frac{\cancel{(s+2)}}{\cancel{(s+2)}(s+\frac{1}{2})}\bigg|_{s=-2} = \frac{1}{-1.5} \qquad B = \frac{\cancel{(s+\frac{1}{2})}}{(s+2)\cancel{(s+\frac{1}{2})}}\bigg|_{s=-\frac{1}{2}} = \frac{1}{1.5}$$

$$v_o(t) = \frac{15}{2}\,\frac{1}{1.5}\left(-e^{-2t} + e^{-\frac{1}{2}t}\right)u(t) = 5\left(e^{-\frac{1}{2}t} - e^{-2t}\right)u(t)$$




$$5\left(e^{-\frac{1}{2}t} - e^{-2t}\right)u(t)$$

Problem

**4.10** Determine $i_L(t)$ in the circuit of **Fig. P4.10** for $t \geq 0$ given that $R = 3.5\ \Omega$, $L = 0.5$ H, and $C = 0.2$ F.

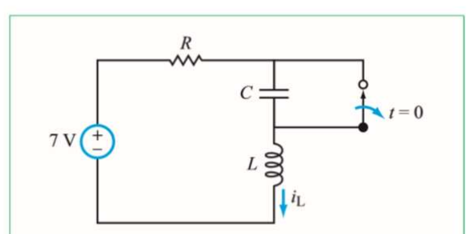



**Figure P4.10:** Circuit for Problem 4.10.

For $t > 0$, $\quad 7 = R\,i_L(t) + \frac{1}{C}\int_0^t i_L(\tau)\,d\tau + L\frac{di_L(t)}{dt}$

$\frac{d}{dt}$ on both sides $\Rightarrow$

$$7\delta(t) = R\frac{di_L(t)}{dt} + \frac{1}{C} i_L(t) + L\frac{d^2 i_L(t)}{dt^2}$$

$\downarrow \mathcal{L}$

$$7 = R\left[sI_L(s) - i_L(0^-)\right] + \frac{1}{C}I_L(s) + L\left[s^2 I_L(s) - s\underbrace{i_L(0)}_{=2} - i_L'(0)\right]$$

$$= \left[Rs + \frac{1}{C} + Ls^2 - 2s\right]I_L(s) - R\,i_L(0^-)$$

$i_L(0) =$, $i_L'(0) = 0$

$$\Rightarrow I_L(s) = \frac{7 + R\,i_L(0)}{Ls^2 + (R-2)s + 5}$$

$$= \frac{7 + R\,i_L(0^-)}{0.5s^2 + 1.5s + 5}$$

$(L = 0.5,\ C = 0.2,\ R = 3.5)$

$$= \frac{14 + x}{s^2 + 3s + 10} = \frac{14 + x}{\left(s + \frac{3}{2}\right)^2 + \frac{31}{4}}$$

$$= (14 + x)\frac{1}{\left(s + \frac{3}{2} + j\frac{\sqrt{31}}{2}\right)\left(s + \frac{3}{2} - j\frac{\sqrt{31}}{2}\right)}$$

$$= (14 + x)\left(\frac{A}{s + \frac{3}{2} + j\frac{\sqrt{31}}{2}} + \frac{B}{s + \frac{3}{2} - j\frac{\sqrt{31}}{2}}\right)$$

$$A = \frac{1}{-j2\left(\frac{\sqrt{31}}{2}\right)} \qquad B = \frac{1}{2j\left(\frac{\sqrt{31}}{2}\right)}$$

$$\Rightarrow i_L(t) = 14 \times \frac{2}{\sqrt{31}} e^{-\frac{3}{2}t}\left(\frac{e^{-j\frac{\sqrt{31}}{2}t}}{-j2} + \frac{e^{+j\frac{\sqrt{31}}{2}t}}{+j2}\right)u(t)$$

$$= \frac{28}{\sqrt{31}} e^{-\frac{3}{2}t} \cdot \sin\frac{\sqrt{31}}{2}t \cdot u(t)$$

Temp. control system

4-9 TEMPERATURE CONTROL SYSTEM 167

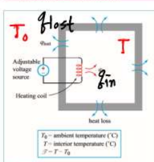



Figure 4-30: Heating coil used to raise the temperature of the interior space of a container relative to the temperature of its outside surroundings, $T_0$.

## 4-9 Temperature Control System

The box shown in Fig. 4-30 represents a container whose initial interior temperature is the same as the ambient temperature $T_0$ of the exterior space surrounding the container. The box might contain a bacterial culture whose growth rate depends on its temperature, a chemical compound whose synthesis requires careful temperature control, or any space whose temperature is to be monitored and controlled. The goal is to design an actuator system that uses an adjustable heat source affixed to an interior wall of the container to raise the container's interior temperature from $T_0$ to a selectable final value $T_\infty$ and to do so fairly quickly. We will examine two system configurations for realizing the stated goal: one without feedback (open-loop mode) and another with feedback (closed-loop mode). Before we do so, however, we will derive the LCCDE relating the heat flow supplied by the heater to the temperature of the container's interior, $T(t)$.

### 4-9.1 Heat Transfer Model

The heat transfer quantities of interest are the following:

- $T_0$ = ambient temperature of the box's exterior space, and also the initial temperature of its interior space.
- $T_\infty$ = the final temperature to which the interior space will be raised (selected by the operator).
- $\Delta T = T_\infty - T_0$ = desired temperature rise.
- $T(t)$ = temperature of the container's interior space as a function of time with $T(0) = T_0$ and $T(\infty) = T_\infty$.
- $\mathcal{T}(t) = T(t) - T_0$ = temperature of container's interior *relative* to the ambient temperature. That is, $\mathcal{T}(t)$ is a ***temperature deviation*** with initial and final values $\mathcal{T}(0) = 0$ and $\mathcal{T}(\infty) = T_\infty - T_0$.
- $Q$ = ***enthalpy*** (heat content in joules) of the interior space of the container (including its contents), defined *relative* to that of the exterior space. That is, $Q = 0$ represents the equilibrium condition when $T = T_0$ or, equivalently, $\mathcal{T} = 0$.
- $C$ = ***heat capacity*** of the container's interior space, measured in joules/°C.
- $R$ = thermal resistance of the interface (walls) between the container's interior and exterior measured in °C per watt (°C/W).
- $q = \frac{dQ}{dt}$ = the rate of heat flow in joules/s or, equivalently, watts (W).

The amount of heat required to raise the temperature of the container's interior by $\mathcal{T}$ is

$$Q = C\mathcal{T}. \qquad (4.114)$$

The rate of heat flow $q$ is analogous to electric current $i$ in electric circuits. For the scenario depicted in Fig. 4-31:

- $q_{in}$ = rate of heat flow supplied by the source.
- $q_{lost}$ = rate of heat flow loss to the outside through the walls.
- $q_{abs}$ = rate of heat flow absorbed by the air and contents of the container's interior, raising their temperature.

For an interface with thermal resistance $R$,

$$q_{lost} = \frac{T - T_0}{R} = \frac{\mathcal{T}}{R} \longleftrightarrow i = \frac{v}{R}. \qquad (4.115)$$

In an electrothermal analogue, temperature deviation $\mathcal{T}$ is equivalent to voltage $v$ (referenced to ground) in an electric circuit. Hence, Eq. (4.115) is the analogue of Ohm's law.

$\hat{T} = T - T_0$

$Q$ = enthalpy of container

$C$ = heat capacity

$R$ = thermal resistance

$Q = C\hat{T}$

$q_{in}$ = rate of heat flow

$q_{lost} = \frac{\hat{T}}{R}$

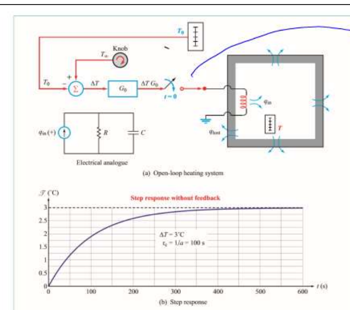



Figure 4-31: Heating system (a) configuration (no feedback) and (b) associated step response for a selected temperature rise of 3 °C.

Differentiating Eq. (4.114) shows that the rate of heat flow absorbed by the container's interior is equal to the increase of its heat content:

$$q_{abs} = \frac{dQ}{dT} = C\frac{d\mathcal{T}}{dt} \longleftrightarrow i = C\frac{dv}{dt}, \qquad (4.116)$$

which is analogous to the $i$-$v$ relationship for a capacitor. Conservation of energy dictates that, analogous to KCL,

$$q_{abs} + q_{lost} = q_{in}$$

or, equivalently,

$$\boxed{C\frac{d\mathcal{T}}{dt} + \frac{\mathcal{T}}{R} = q_{in}(t).} \qquad (4.117)$$

The electrical analogue of the *thermal circuit* consists of a parallel RC circuit connected to a current source $q_{in}(t)$, as shown in Fig. 4-31(a).

Heat Amount $= \Delta T\, q_0\, u(t)$

$\Updownarrow$

$\frac{\Delta T\, q_0}{s}$

$$C\frac{d\hat{T}}{dt} + \frac{\hat{T}}{R} = q_{in}$$

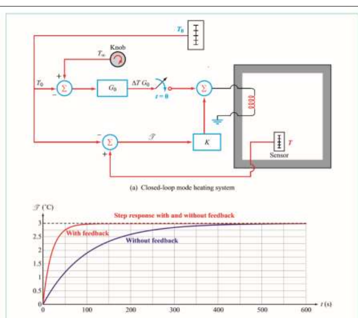



Figure 4-32: Heating system (a) configuration (with feedback) and (b) associated step response for a selected temperature rise of 3 °C.

The Laplace transform of $x(t)$ is

$$\mathbf{X}(\mathbf{s}) = \left(\frac{\Delta T\, G_0}{C}\right)\frac{1}{\mathbf{s}},$$

and the output step response is

$$\mathbf{Y}(\mathbf{s}) = \mathbf{Q}(\mathbf{s})\,\mathbf{X}(\mathbf{s}) = \left(\frac{\Delta T\, G_0}{C}\right)\frac{1}{\mathbf{s}(\mathbf{s}+b)}.$$

Partial fraction expansion leads to

$$\mathbf{Y}(\mathbf{s}) = \left(\frac{\Delta T\, G_0}{bC}\right)\left[\frac{1}{\mathbf{s}} - \frac{1}{\mathbf{s}+b}\right], \qquad (4.123)$$

which has the inverse Laplace transform

$$T(t) = \frac{\Delta T\, G_0}{bC}\left[1 - e^{-bt}\right]u(t). \qquad (4.124)$$

$$q_{in}(t) = C\frac{d\hat{T}}{dt} + \frac{\hat{T}}{R}$$

$\downarrow \mathcal{L}$

$$\frac{\Delta T\, q_0}{s} = \left(Cs + \frac{1}{R}\right)\hat{T}(s)$$

$$\hat{T}(s) = \frac{\Delta T\, q_0}{s\left(Cs + \frac{1}{R}\right)}$$

$$= \frac{\Delta T\, q_0}{C}\frac{1}{s\left(s + \frac{1}{RC}\right)}$$

$$\hat{T}(s) = \frac{\Delta T\, q_0}{C}\frac{1}{s\left(s + \frac{1}{RC}\right)} = \left[\frac{\Delta T\, q_0}{s}\right]\overbrace{\left[\frac{1}{C\left(s + \frac{1}{RC}\right)}\right]}^{H(s)}$$

$$= \frac{\Delta T\, q_0}{C}\frac{1}{s(s+b)} = \frac{\Delta T\, q_0}{C}\left[\frac{\frac{1}{b}}{s} + \frac{-\frac{1}{b}}{s+b}\right]$$

let $b = \frac{1}{RC}$

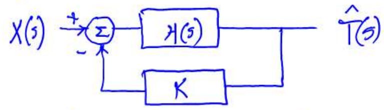
$$= \frac{\Delta T\, q_0}{Cb}\left[\frac{1}{s} - \frac{1}{s+b}\right]$$

$\downarrow \mathcal{L}^{-1}$

$$\hat{T}(t) = \frac{\Delta T\, q_0}{C\left(\frac{1}{RC}\right)}\left(1 - e^{-\frac{1}{RC}t}\right)u(t)$$

$$= \Delta T\, R\, q_0\left(1 - e^{-\frac{1}{RC}t}\right)u(t)$$

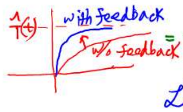
With feedback, temperature rises faster:




$$\hat{T}(s) = \left[X(s) - K\,\hat{T}(s)\right]H(s)$$

$$\hat{T}(s)\left(1 + K\,H(s)\right) = H(s)\,X(s)$$

$$\hat{T}(s) = \frac{H(s)}{1 + K\,H(s)}\,X(s)\left(= \frac{\Delta T\,q_0 R}{s}\right)$$




$$= \frac{\left(\frac{1}{s+b}\right)X(s)}{1 + K\frac{1}{s+b}} = \frac{1}{s+(b+K)}\,\frac{\Delta T\,q_0 R}{s}$$

$$\mathcal{L}^{-1} \longrightarrow \hat{T}(t) = \Delta T\,q_0 R\left(1 - e^{-(b+K)t}\right)u(t)$$