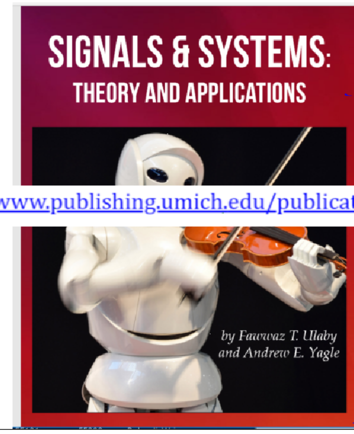
ECE 103 Lecture 25, Nov. 30, 2018

No more quiz. Your best 6 out of 8 will consistute 20% of your final score

We will not cover double-side Laplace transform.

Instead we will go over the important subjects using the Ulaby + Yagle eBook.



https://www.publishing.umich.edu/publications/ee/

$$H(s)=K\frac{N(s)}{D(s)}=K\frac{s^m+b_1s^{m-1}+b_2s^{m-2}+\cdots+b_{m-1}s+b_m}{s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_{n-1}s+a_n}$$

when $n \geq m$

$H(s)$ can be written as

$$H(s)=K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}=K\frac{\prod_{i=1}^{m}(s+x_i)}{\prod_{j=1}^{n}(s+d_j)}$$

e.g. $$H_1(s)=10\frac{s^2+4}{(s+1)(s^2+9)}=10\frac{\overbrace{s^2+0s+4}^{N(s)}}{\underbrace{s^3+s^2+9s+9}_{D(s)}}$$

$$=10\frac{(s+j2)(s-j2)}{(s+1)(s+j3)(s-j3)}=10\frac{(s-(-j2))(s-j2)}{(s-(-1))(s-(-j3))(s-j3)}$$

$$=10\frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)(s-p_3)}$$

$z_1=-j2,\ z_2=j2$

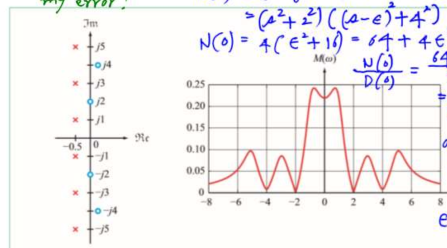
$p_1=-1,\ p_2=-j3,\ p_3=j3$

Daniel caught my error!

$N(s)=(s+j2)(s-j2)(s-\epsilon+j4)(s-\epsilon-j4)$

$=(s^2+2^2)((s-\epsilon)^2+4^2)$

$N(0)=4(\epsilon^2+16)=64+4\epsilon^2$



(a) Poles and zeros (b) Magnitude spectrum

Figure 6-25: Magnitude spectrum of a frequency response composed of 2 pairs of conjugate zeros and 3 pairs of conjugate poles (Example 6-6).

$\frac{N(0)}{D(0)}=\frac{64+4\epsilon^2}{\cancel{315.31}\ 291.95}$

$=\cancel{0.203}\ 0.219+\frac{4\epsilon^2}{\cancel{315.31}\ 291.35}$

$0.225-\cancel{0.203}\ 0.219$

$=\frac{4\epsilon^2}{\cancel{315.31}\ 291.95}$

$\epsilon=\sqrt{\frac{\cancel{0.022}\ 0.006\times\cancel{315.31}\ 291.95}{4}}$

$=\cancel{1.32}\ 0.66$

$D(s)=(s+0.5+j1)(s+0.5-j1)(s+0.5+j3)(s+0.5-j3)(s+0.5+j5)(s+0.5-j5)$

$=((s+0.5)^2+1)((s+0.5)^2+3^2)((s+0.5)^2+5^2)$

$D(0)=1.25\times9.25\times25.25$

$=\cancel{315.31}\ 291.95$

$V_i(s) \rightarrow \boxed{H(s)} \rightarrow V_o(s)$

To derive $H(s)$, set all i.c.s to zero; system/transfer functions do not depend on i.c.s.

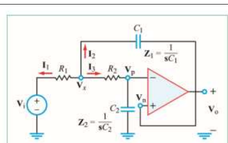
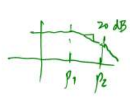


Figure 6-43: Sallen-Key circuit.



(a) Obtain the transfer function $\mathbf{H}(s)=\mathbf{V}_o(s)/\mathbf{V}_i(s)$.

~~(b) Obtain the transfer function of a second-order Butterworth~~ filter with cutoff frequency $\omega_c = 10^3$ rad/s and arbitrary dc gain **C**.

(c) ~~Match the transfer functions in (a) and (b) to specify the~~ values of R_1, R_2, C_1, and C_2.

Solution:

(a) At node $\mathbf{V}_x$, KCL gives

$$\frac{\mathbf{V}_x-\mathbf{V}_i}{R_1}+\frac{\mathbf{V}_x-\mathbf{V}_p}{R_2}+\frac{\mathbf{V}_x-\mathbf{V}_o}{\mathbf{Z}_1}=0, \quad (6.102)$$

and since $\mathbf{V}_o=\mathbf{V}_n=\mathbf{V}_p$, voltage division gives

$$\mathbf{V}_x=\mathbf{V}_o\left(\frac{R_2+\mathbf{Z}_2}{\mathbf{Z}_2}\right). \quad (6.103)$$

The combination of the two equations leads to

$$\mathbf{H}(s)=\frac{\frac{1}{R_1R_2C_1C_2}}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}. \quad (6.104)$$

$(x+y)^2=x^2+2xy+y^2$

$-)\ (x-y)^2=x^2-2xy+y^2$

$4xy$

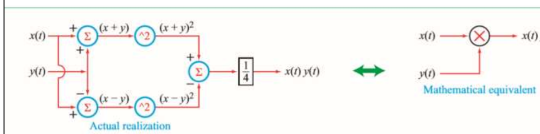
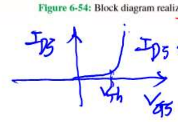
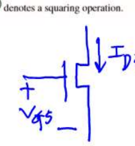


Figure 6-54: Block diagram realization of the product $x(t)\ y(t)$. The symbol ^2 denotes a squaring operation.



$I_{DS}=\frac{k}{2}(V_{GS}-V_{th})^2$

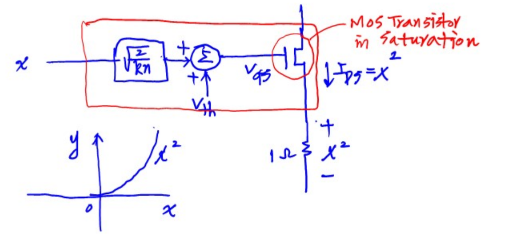
$=0 \quad V_{GS}<V_{th}$



If $x = \sqrt{\frac{k_n}{2}}(V_{GS} - V_{th})$ or $\boxed{V_{GS} = \sqrt{\frac{2}{k_n}}\, x + V_{th}}$

$x^2 = \frac{k_n}{2}(V_{GS} - V_{th})^2$

and $I_{DS} = \frac{k_n}{2}(V_{GS} - V_{th})^2 = x^2$



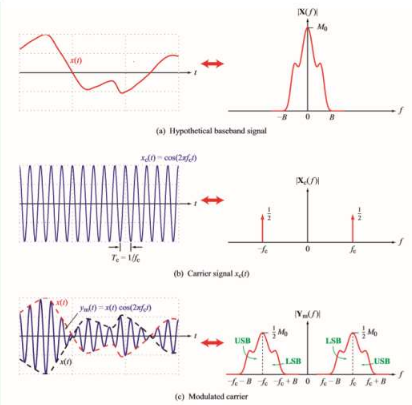


Figure 6-56: Baseband signal $x(t)$, carrier signal $x_c(t)$, and modulated carrier $y_m(t)$.

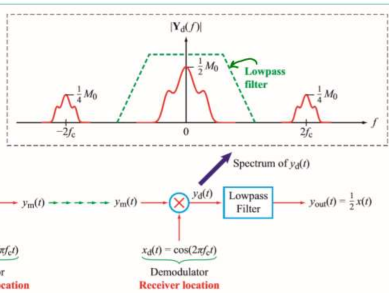
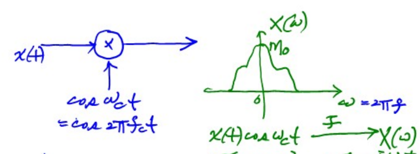


Figure 6-57: The DSB demodulator uses a replica of the carrier signal, followed by a lowpass filter, to recover the baseband signal $x(t)$.



$x(t)\cos\omega_c t \xrightarrow{\mathcal{F}} X(\omega) * \mathcal{F}[\cos\omega_c t]$

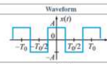
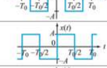
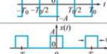
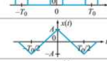
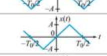
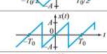
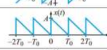
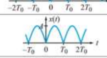
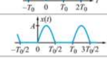
$\mathcal{F}[\cos\omega_c t] = \mathcal{F}\left[\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right]$

$= \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$

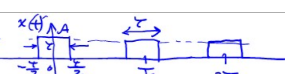
$\mathcal{F}[e^{j\omega_c t}] = 2\pi\delta(\omega - \omega_c)$

$X(\omega) * \mathcal{F}[\cos\omega_c t] = \pi[X(\omega - \omega_c) + X(\omega + \omega_c)]$

Table 5-4: Fourier series expressions for a select set of periodic waveforms.

	Waveform	Fourier Series
1. Square Wave		$x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{2n\pi t}{T_0}\right)$
2. Time-Shifted Square Wave		$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{4A}{n\pi}\sin\left(\frac{2n\pi t}{T_0}\right)$
3. Pulse Train		$x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi}\sin\left(\frac{n\pi\tau}{T_0}\right)\cos\left(\frac{2n\pi t}{T_0}\right)$
4. Triangular Wave		$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{8A}{n^2\pi^2}\cos\left(\frac{2n\pi t}{T_0}\right)$
5. Shifted Triangular Wave		$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{8A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{2n\pi t}{T_0}\right)$
6. Sawtooth		$x(t) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2A}{n\pi}\sin\left(\frac{2n\pi t}{T_0}\right)$
7. Backward Sawtooth		$x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{n\pi}\sin\left(\frac{2n\pi t}{T_0}\right)$
8. Full-Wave Rectified Sinusoid		$x(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty} \frac{4A}{\pi(1-4n^2)}\cos\left(\frac{2n\pi t}{T_0}\right)$
9. Half-Wave Rectified Sinusoid		$x(t) = \frac{A}{\pi} + \frac{A}{2}\sin\left(\frac{2\pi t}{T_0}\right) + \sum_{\substack{n=2 \\ n=\text{even}}}^{\infty} \frac{2A}{\pi(1-n^2)}\cos\left(\frac{2n\pi t}{T_0}\right)$

For



periodic $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, $\omega_0 = \frac{2\pi}{T_0}$

$c_k = \frac{1}{T_0}\int_{period} x(t) e^{-jk\omega_0 t} dt$

$c_0 = \frac{1}{T_0}\int_{period} x(t)\, dt = \frac{A\tau}{T_0}$ = DC average

$c_k = \frac{1}{T_0}\int_{period} x(t) e^{-jk\omega_0 t} dt$

$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A e^{-jk\omega_0 t} dt = \frac{1}{T_0} \frac{A}{-jk\omega_0} e^{-jk\omega_0 t}\Big]_{-\tau/2}^{\tau/2}$

$= \frac{1}{T_0} \frac{A}{-jk\omega_0}\left[e^{-jk\omega_0 \frac{\tau}{2}} - e^{-jk\omega_0(-\frac{\tau}{2})}\right]$

$= \frac{1}{T_0} \frac{2A}{k\omega_0} \frac{e^{+jk\omega_0 \tau/2} - e^{-jk\omega_0(\tau/2)}}{2j}$

$= \frac{1}{T_0} \frac{2A}{k\omega_0} \sin k\omega_0 \tau/2 = \frac{A}{k\pi} \sin k\omega_0 \frac{\tau}{2}$

$$x(t) = \frac{A\tau}{T_0} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\frac{A}{k\pi} \sin\left(k\omega_0 \frac{\tau}{2}\right)\right] e^{-jk\omega_0 t}$$

$$= \frac{A\tau}{T_0} + \sum_{k=1}^{\infty} \left(\frac{A}{k\pi} \sin k\omega_0 \frac{\tau}{2}\right) \underbrace{\left(\frac{e^{-jk\omega_0 t} + e^{-j(-k)\omega_0 t}}{2}\right)}_{= \cos k\omega_0 t \; = \cos k \frac{2\pi}{T_0} t} 2$$

$$\sin k \frac{2\pi}{T_0} \frac{\tau}{2} = \sin k\pi \frac{\tau}{T_0}$$

$$= \frac{A\tau}{T_0} + \sum_{k=1}^{\infty} \frac{2A}{k\pi} \sin\left(k\pi \frac{\tau}{T_0}\right) \cos\left(2\pi k \frac{t}{T_0}\right)$$

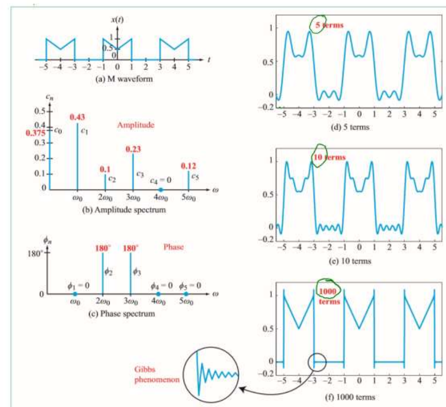


Figure 5-7: Plots for Example 5-5.

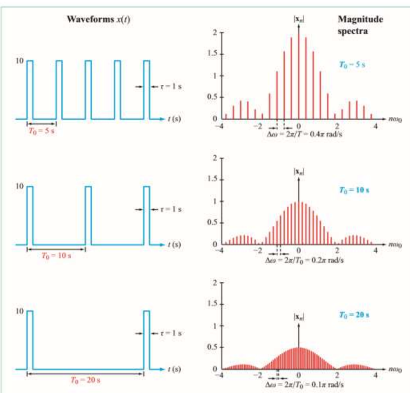
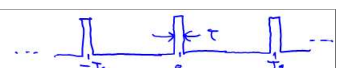


Figure 5-12: Line spectra for pulse trains with $T_0/\tau = 5$, 10, and 20.

Table 5-6: Examples of Fourier transform pairs. Note that constant $a \geq 0$.

	$x(t)$	$\mathbf{X}(\omega) = \mathcal{F}[x(t)]$	$\|\mathbf{X}(\omega)\|$
	BASIC FUNCTIONS		
1.	$\delta(t)$	1	
1a.	$\delta(t - t_0)$	$e^{-j\omega t_0}$	
2.	1	$2\pi\,\delta(\omega)$	
3.	$u(t)$	$\pi\,\delta(\omega) + 1/j\omega$	
4.	$\mathrm{sgn}(t)$	$2/j\omega$	
5.	$\mathrm{rect}(t/\tau)$	$\tau\,\mathrm{sinc}(\omega\tau/2)$	
6.	$\dfrac{e^{-t^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}$	$e^{-\omega^2\sigma^2/2}$	
7a.	$e^{-at}\,u(t)$	$1/(a + j\omega)$	
7b.	$e^{at}\,u(-t)$	$1/(a - j\omega)$	
8.	$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
9.	$\sin\omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$	
	ADDITIONAL FUNCTIONS		
10.	$e^{j\omega_0 t}$	$2\pi\,\delta(\omega - \omega_0)$	
11.	$te^{-at}\,u(t)$	$1/(a + j\omega)^2$	
12a.	$[e^{-at}\sin\omega_0 t]\,u(t)$	$\omega_0/[(a + j\omega)^2 + \omega_0^2]$	
12b.	$[\sin\omega_0 t]\,u(t)$	$(\pi/2j)[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + [\omega_0^2/(\omega_0^2 - \omega^2)]$	
13a.	$[e^{-at}\cos\omega_0 t]\,u(t)$	$(a + j\omega)/[(a + j\omega)^2 + \omega_0^2]$	
13b.	$[\cos\omega_0 t]\,u(t)$	$(\pi/2)[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + [j\omega/(\omega_0^2 - \omega^2)]$	



According to **Fig. 6-83**, for $f_u/B = 1.5$, the sampling rate should be

$$f_s = 3B = 3 \times 100 = 300 \text{ Hz}.$$

Per Eq. (6.158), the spectrum of the sampled signal is given by

$$\mathbf{X}_s(f) = f_s \sum_{n=-\infty}^{\infty} \mathbf{X}(f - nf_s).$$

Spectrum $\mathbf{X}_s(t)$ consists of the original spectrum, plus an infinite number of duplicates, shifted to the right by nf_s for $n > 0$ and to the left by $|nf_s|$ for $n < 0$. All spectra are scaled by a multiplicative factor of f_s. **Figure 6-86(b)** displays the spectra for $n = 0$ and ± 1. We note that the spectra do not overlap, which means that the original signal can be reconstructed by passing spectrum $\mathbf{X}_s(f)$ through a lowpass filter with a spectral response that extends between -150 Hz and $+150$ Hz.

6-13.14 Practical Aspects of Sampling

So far, all our discussions of signal sampling assumed the availability of an ideal impulse train composed of impulses. In practice, the sampling is performed by finite-duration pulses that may resemble rectangular or Gaussian waveforms. **Figure 6-84** depicts the sampling operation for a signal $x(t)$, with the sampling generated by a train of rectangular pulses given by

$$p_{T_s}(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t - nT_s}{\tau}\right), \quad \mathrm{rect}\left(\left.t - nT_s\right|\tau\right)$$

where T_s is the sampling interval and τ is the pulse width. From **Table 5-4**, the Fourier series representation of the pulse train is

$$p_{T_s}(t) = \frac{\tau}{T_s} + \sum_{m=1}^{\infty} \frac{2}{m\pi} \sin\left(\frac{m\pi\tau}{T_s}\right) \cos\left(\frac{2m\pi\tau}{T_s}\right). \quad (6.166)$$

The signal sampled by the pulse train is

$$x_s(t) = x(t)\, p_{T_s}(t)$$
$$= \frac{\tau}{T_s}\, x(t) + \sum_{m=1}^{\infty} \frac{2}{m\pi}\, x(t) \sin\left(\frac{m\pi\tau}{T_s}\right) \cos\left(\frac{2m\pi t}{T_s}\right). \quad (6.167)$$

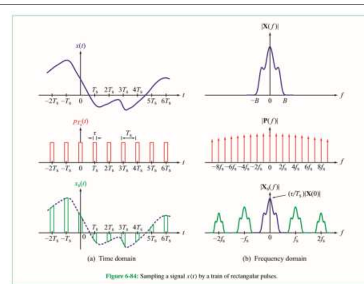


Figure 6-84: Sampling a signal $x(t)$ by a train of rectangular pulses.

which can be cast in the form

$$x_s(t) = A_0\, x(t) + A_1\, x(t) \cos(2\pi f_s t) + A_2\, x(t) \cos(4\pi f_s t) + \cdots, \quad (6.168)$$

with

$$f_s = \frac{1}{T_s}, \qquad A_0 = \frac{\tau}{T_s},$$
$$A_1 = \frac{2}{\pi} \sin(\pi f_s \tau), \qquad A_2 = \frac{1}{\pi} \sin(2\pi f_s \tau), \ldots .$$

The sequence given by Eq. (6.168) consists of a dc term, $A_0\, x(t)$, and a sum of sinusoids at frequency f_s and its harmonics. The Fourier transform of the A_0 term is $A_0\, \mathbf{X}(f)$, where $\mathbf{X}(f)$ is the transform of $x(t)$. It is represented by the central spectrum of $\mathbf{X}_s(f)$ in **Fig. 6-84**. The cosine terms in Eq. (6.168) generate image spectra centered at $\pm f_s$ and its harmonics, but their amplitudes are modified by the values of A_1, A_2, etc.

Signal $x(t)$ can be reconstructed from $x_s(t)$ by lowpass filtering it, just as was done earlier with the ideal impulse sampling. The Shannon sampling requirement that f_s should be greater than $2B$ still holds.

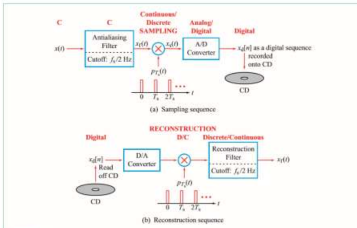


Figure 6-85: DSP system with C/D and D/C to convert between continuous time and discrete time, and A/D and D/A to convert between analog and digital.

By way of an example, we show in Fig. 6-85 a block diagram of a typical DSP system that uses a compact disc (CD) for storage of digital data. The sampling part of the process starts with a continuous-time signal $x(t)$. After bandlimiting the signal to $f_s/2$ by an antialiasing filter, the filtered signal $x_f(t)$ is sampled at a rate f_s and then converted into a digital sequence $x_d[n]$ that gets recorded onto the CD.

Reconstruction performs the reverse sequence, starting with $x_d[n]$ as read off of the CD and concluding in $x_f(t)$, the bandlimited version of the original signal.

Concept Question 6-19: Does sampling a signal at exactly the Nyquist rate guarantee that it can be reconstructed from its discrete samples? (See S²)

Concept Question 6-20: What is signal aliasing? What causes it? How can it be avoided? (See S²)

Concept Question 6-21: If brick-wall lowpass filters are used in connection with a signal bandlimited to f_{max} and sampled at f_s, what should the filter's cutoff frequency be when used as (a) an anti-aliasing filter and (b) a reconstruction filter? (See S²)

Exercise 6-18: What is the Nyquist sampling rate for a signal bandlimited to 5 kHz?

Answer: 10 kHz. (See S²)

Exercise 6-19: A 500 Hz sinusoid is sampled at 900 Hz. No anti-alias filter is used. What is the frequency of the reconstructed sinusoid?

Answer: 400 Hz. (See S²)

Another example for finding $H(s)$ or $H(\omega)$

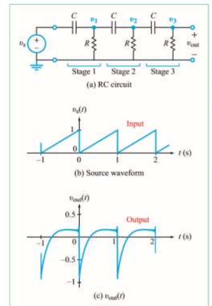
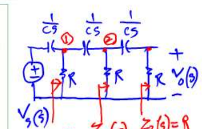


Figure 5-10: Circuit and plots for Example 5-7.



$Z_1(s)$ $Z_2(s)$ $Z_3(s) = R$

$$V_1(s) = \frac{Z_1(s)}{\frac{1}{Cs} + Z_1(s)} V_s(s)$$

$$V_2(s) = V_1(s) \frac{Z_2(s)}{\frac{1}{Cs} + Z_2(s)}$$

$$V_o(s) = V_2(s) \frac{R}{\frac{1}{Cs} + R}$$

Example 5-8: Three-Stage RC Circuit

Application of KCL to the three-stage RC circuit of Fig. 5-10(a) and defining $x = \omega RC$ leads to

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_s} = \frac{x^3}{(x^3 - 5x) + j(1 - 6x^2)} .$$