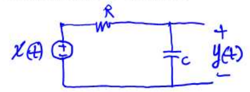
ECE 103 Lecture 26 Dec 3, 2018

Quiz 8 Average = 7.40
σ = 2.21

Example 5-7: RC Circuit

Determine $\upsilon_{\text{out}}(t)$ when the circuit in **Fig. 5-9(a)** is excited by the voltage waveform shown in **Fig. 5-9(b)**. The element values are $R = 20\ \text{k}\Omega$ and $C = 0.1$ mF.



Today Dec 3 (M)

Dec 5 (W) course review

Dec 7 (F) solving problems by TA Azzam

For periodic $x(t) = x(t+T_0)$, $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$, $\omega_0 = 2\pi/T_0$

$c_k = a_k - jb_k$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t)\left[\cos k\omega_0 t - j\sin k\omega_0 t\right] dt$$

$$= \underbrace{\frac{1}{T_0}\int_{T_0} x(t)\cos k\omega_0 t\, dt}_{a_k} - j\underbrace{\frac{1}{T_0}\int_{T_0} x(t)\sin k\omega_0 t\, dt}_{b_k}$$

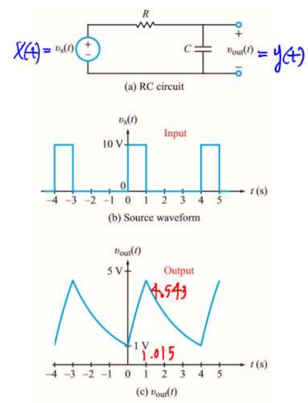


Figure 5-9: Circuit response to periodic pulses.

RC = 20 kΩ × 0.1 mF
= 2

Solution:

$T_0 = 4$

Step 1: The period of $\upsilon_s(t)$ is 4 s. Hence, $\omega_0 = 2\pi/4 = \pi/2$ rad/s, and by Eq. (5.28), we have

$$a_0 = \frac{1}{T}\int_0^T \upsilon_s(t)\, dt = \frac{1}{4}\int_0^1 10\, dt = 2.5\ \text{V},$$

$$a_n = \frac{2}{4}\int_0^1 10\cos\frac{n\pi}{2}t\, dt = \frac{10}{n\pi}\sin\frac{n\pi}{2}\ \text{V},$$

$$5\int_0^1 \frac{2}{n\pi}\, d\left(\sin\frac{n\pi}{2}\right)$$

$$b_n = \frac{2}{4}\int_0^1 10\sin\frac{n\pi}{2}t\, dt = \frac{10}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\ \text{V},$$

and

$$\hat{c}_n = c_n\angle\phi_n = a_n - jb_n = \frac{10}{n\pi}\left[\sin\frac{n\pi}{2} - j\left(1 - \cos\frac{n\pi}{2}\right)\right].$$

The values of $c_n\angle\phi_n$ for the first four terms are

$$\hat{c}_1 = c_1\angle\phi_1 = \frac{10\sqrt{2}}{\pi}\angle{-45^\circ},$$

$$\hat{c}_2 = c_2\angle\phi_2 = \frac{10}{\pi}\angle{-90^\circ},$$

$$\hat{c}_3 = c_3\angle\phi_3 = \frac{10\sqrt{2}}{3\pi}\angle{-135^\circ},$$

and

$\hat{C}_4 = \quad c_4 \underline{/\phi_4} = 0.$

Step 2: In the phasor domain, the impedance of a capacitor is $\mathbf{Z}_C = 1/(j\omega C)$. By voltage division, the generic phasor-domain transfer function of the circuit is

$$\begin{aligned}
\mathbf{H}(\omega) &= \mathbf{V}_{out} \qquad (\text{with } \mathbf{V}_s = 1) \quad v_s(t) = \delta(t) \\
&= \frac{1}{1 + j\omega RC} \\
&= \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}\, e^{-j\tan^{-1}(\omega RC)} \\
&= \frac{1}{\sqrt{1+4\omega^2}}\, e^{-j\tan^{-1}(2\omega)},
\end{aligned}$$

$V_s(\omega)$ — $\boxed{H(\omega)}$ — $V_o(\omega)$

$\rightarrow \boxed{H(\omega)} \rightarrow$

$\hat{C}_n^{y}$: $\boxed{\begin{matrix} |\hat{C}_n|\,|H(\omega)| \\ \angle \hat{C}_n + \angle H(\omega) \end{matrix}}$

where we used $RC = 2 \times 10^4 \times 10^{-4} = 2$ s.

Step 3: The time-domain output voltage is

$$v_{out}(t) = 2.5 + \sum_{n=1}^{\infty} \Re\mathfrak{e} \left\{ c_n \frac{1}{\sqrt{1+4n^2\omega_0^2}}\, e^{j[n\omega_0 t + \phi_n - \tan^{-1}(2n\omega_0)]} \right\}.$$

Using the values of $c_n \underline{/\phi_n}$ determined earlier for the first four terms and replacing ω_0 with its numerical value of $\pi/2$ rad/s, the expression becomes

$$\begin{aligned}
v_{out}(t) &= 2.5 \\
&+ \frac{10\sqrt{2}}{\pi\sqrt{1+\pi^2}} \cos\left[\frac{\pi t}{2} - 45^\circ - \tan^{-1}(\pi)\right] \\
&+ \frac{10}{\pi\sqrt{1+4\pi^2}} \cos[\pi t - 90^\circ - \tan^{-1}(2\pi)] \\
&+ \frac{10\sqrt{2}}{3\pi\sqrt{1+9\pi^2}} \cos\left[\frac{3\pi t}{2} - 135^\circ - \tan^{-1}(3\pi)\right] \cdots \\
&= 2.5 + 1.37\cos\left(\frac{\pi t}{2} - 117^\circ\right) + 0.5\cos(\pi t - 171^\circ) \\
&+ 0.16\cos\left(\frac{3\pi t}{2} + 141^\circ\right) \cdots \text{ V}
\end{aligned}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{C}_n e^{+jn\omega_0 t}$$

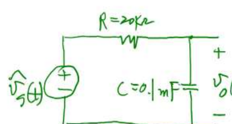
$$X(\omega) = \sum_{n=-\infty}^{\infty} \hat{C}_n \, 2\pi\, \delta(\omega - n\omega_0)$$

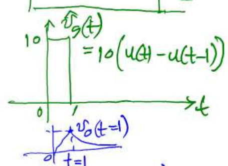
$$V_o(\omega) = X(\omega) H(\omega)$$

$$= \sum_{n=-\infty}^{\infty} \hat{C}_n \, 2\pi\, H(n\omega_0)\, \delta(\omega - n\omega_0)$$

$$v_o(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\phi_n} |H(n\omega_0)|\, e^{j\angle H(n\omega_0)} e^{jn\omega_0 t}$$

$$= C_0 |H(0)| + \sum_{n=1}^{\infty} 2C_n |H(n\omega_0)| \cos(n\omega_0 t + \phi_n + \angle H(n\omega_0))$$





$= 10(u(t) - u(t-1))$

$$\hat{v}_s(t) = RC\frac{dv_o(t)}{dt} + v_o(t)$$

$$\hat{V}_s(s) = RC[sV_o(s) - v_o(0^-)] + V_o(s)$$

$$\hat{V}_s(s) + RCv_o(0^-) = (RCs+1)V_o(s)$$

$$V_o(s) = \frac{\hat{V}_s(s) + RCv_o(0^-)}{RCs+1}$$

$$= \frac{\hat{V}_s(s)/RC + v_o(0^-)}{s + \frac{1}{RC}}$$

$$= \frac{1}{RC}\hat{V}_s(s)\frac{1}{s+\frac{1}{RC}} + v_o(0^-)\frac{1}{s+\frac{1}{RC}}$$

For $RC = 20 \times 10^3 \times 0.1 \times 10^{-3} = 2$,

For $0 < t < 1$ $\quad V_o(s) = \frac{1}{2}\frac{10}{s}\frac{1}{s+0.5} + v_o(0^-)\frac{1}{s+0.5}$

$$v_o(t) = [10(1 - e^{-0.5t}) + v_o(0^-)e^{-0.5t}]\,u(t)$$

At $t = 4$, $\quad v_o(4) = [10(1 - e^{-0.5}) + v_o(0^-)e^{-0.5}]\,e^{-0.5\times(4-1)} = v_o(0^-)$

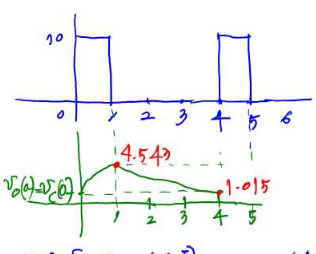
$$\frac{1}{2}\,\frac{10}{s}\,\frac{1}{s+0.5} = 5\left[\frac{2}{s} + \frac{-2}{s+0.5}\right]$$

$$\frac{1}{s(s+0.5)} = \frac{A}{s} + \frac{B}{s+0.5}$$

$$A = \frac{1}{s(s+0.5)} \times s\Big|_{s=0} = \frac{1}{s+0.5}\Big|_{s=0} = 2$$

$$B = \frac{1}{s(s+0.5)}(s+0.5)\Big|_{s=-0.5} = -2$$

$$\rightarrow 10\left[\frac{1}{s} - \frac{1}{s+0.5}\right]$$



$4.550\,e^{-1.5} = 1.015$

$$v_o(4) = [10(1 - e^{-0.5}) + v_o(0^-)e^{-0.5}]\,e^{-1.5} = v_o(0^-)$$

$$10(1 - e^{-0.5}) = v_o(0^-)(e^{1.5} - e^{-0.5})$$

$$v_o(0^-) = \frac{10(1 - e^{-0.5})}{e^{1.5} - e^{-0.5}} = 1.015$$

$$v_o(1) = v_o(0^-)e^{-0.5} + 10(1 - e^{-0.5}) = 0.6156 + 10 \times 0.39347$$

$$= 4.550$$

6.3 For the circuit shown in **Fig. P6.3**, determine (a) the transfer function $\mathbf{H} = \mathbf{V}_o/\mathbf{V}_i$, and (b) the frequency ω_0 at which $\mathbf{H}$ is purely real.

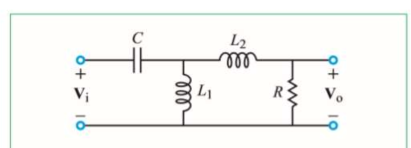
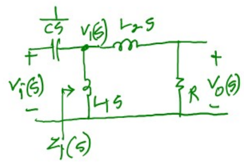


Figure P6.3: Circuit for Problem 6.3.



$$Z_1(s) = L_1 s \parallel (L_2 s + R) = \frac{L_1 s (L_2 s + R)}{L_1 s + (L_2 s + R)}$$

$$= \frac{L_1 L_2 s^2 + L_1 R s}{(L_1 + L_2) s + R}$$

$$V_1(s) = V_i(s) \frac{Z_1(s)}{\frac{1}{Cs} + Z_1(s)}$$

$$= V_i(s) \frac{Z_1(s) Cs}{1 + Z_1(s) Cs} = V_i(s) \frac{(L_1 L_2 s^2 + L_1 R s) Cs}{(L_1 + L_2) s + R + (L_1 L_2 s^2 + L_1 R s) Cs}$$

$$V_o(s) = V_1(s) \frac{R}{L_2 s + R}$$

$$= V_i(s) \frac{(L_1 L_2 s^2 + L_1 R s) Cs}{(L_1 + L_2) s + R + (L_1 L_2 s^2 + L_1 R s) Cs} \quad \frac{R}{L_2 s + R}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{(L_1 L_2 s^2 + L_1 R s) RCs}{[(L_1 + L_2) s + R + (L_1 L_2 s^2 + L_1 R s) Cs](L_2 s + R)}$$

$$= \frac{L_1 RCs^2 (\cancel{L_2 s +} \; R)}{(L_1 L_2 s^3 + L_1 RCs^2 + (L_1 + L_2) s + R)\cancel{(L_2 s + R)}}$$

$$H(\omega) = \frac{L_1 RC(-\omega^2)}{j\omega(-L_1 L_2 \omega^2) - L_1 RC\omega^2 + (L_1 + L_2) j\omega + R}$$

denominator

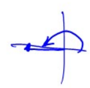
$$j\omega(L_1 + L_2 - L_1 L_2 \omega^2) - L_1 RC\omega^2 + R$$

real when $\omega = \sqrt{\frac{L_1 + L_2}{L_1 L_2}}$

$$H(\omega) = \frac{L_1 RC(-\omega^2)}{j\omega(L_1 + L_2 - L_1 L_2 \omega^2) - L_1 RC\omega^2 + R}$$

$$\angle H(\omega) = \angle \text{numerator} - \angle \text{denominator}$$

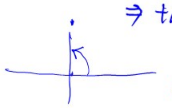
$$\angle \text{numerator} = \angle -(L_1 RC\omega^2) = 180^\circ$$



$$\angle \text{denominator} = \tan^{-1}\left[\frac{(L_1 + L_2 - L_1 L_2 \omega^2)}{R_1 - L_1 RC\omega^2}\right]$$

$\angle H(\omega) = 90^\circ \quad \omega = ?$

$$\Rightarrow \tan^{-1}\left[\frac{L_1 + L_2 - L_1 L_2 \omega^2}{R_1 - L_1 RC\omega^2}\right] \stackrel{?}{=} 90^\circ$$



$$\epsilon = R_1 - L_1 RC\omega^2 \qquad \frac{\epsilon - R_1}{-L_1 RC} = -\cancel{L_1 RC}\omega^2 = \omega^2$$

$\omega = \sqrt{\frac{R_1 - \epsilon}{L_1 RC}}$ for small ϵ