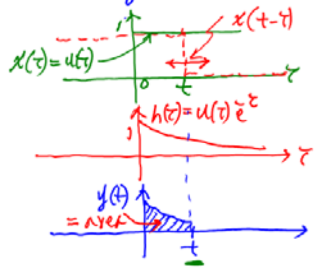
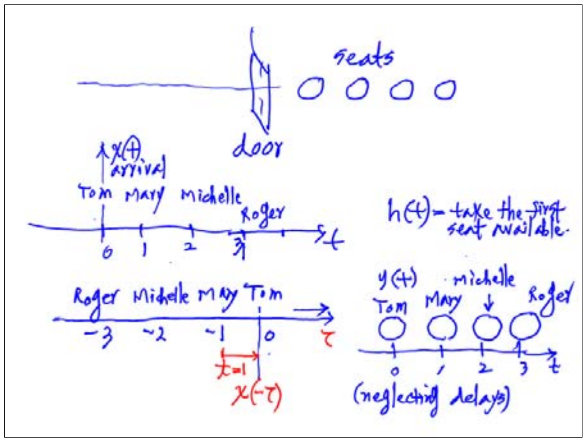
ECE 103 Lecture #6 Oct 10, 2018

Continuation of convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$



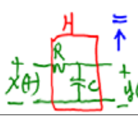
$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

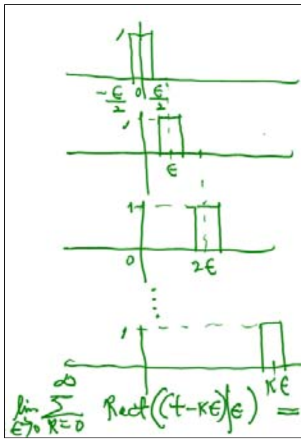


$x(t)=u(t)$ $\qquad y(t) = H(x(t))$

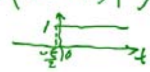
$$x(t) = \lim_{\epsilon\to 0} \sum_{k=0}^{\infty} Rect((t-k\epsilon)|\epsilon) = u(t)$$

$$y(t) = H(x(t)) = H \lim_{\epsilon\to 0} \sum_{k=0}^{\infty} Rect((t-k\epsilon)|\epsilon)$$
$$= \lim_{\epsilon\to 0} \sum_{k=0}^{\infty} \cdot H\, \epsilon\, \frac{Rect((t-k\epsilon)|\epsilon)}{\epsilon}$$
$$= \sum_{k=0}^{\infty} \lim_{\epsilon\to 0} \epsilon \cdot H \underbrace{\left(\frac{Rect((t-k\epsilon)|\epsilon)}{\epsilon}\right)}_{=\delta(t-k\epsilon)}$$
$$\frac{1}{RC} e^{-\frac{1}{RC}(t-k\epsilon)}\, u(t-k\epsilon)$$





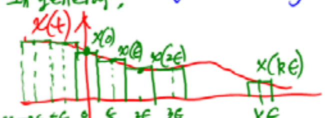
$$\lim_{\epsilon\to 0} \sum_{k=0}^{\infty} Rect((t-k\epsilon)|\epsilon) = u(t)$$



If we set $k\epsilon = \tau$, $\epsilon = d\tau$

$$\rightarrow \int_0^t \frac{1}{RC} e^{-(t-\tau)}\, d\tau$$
$$= \left(1 - e^{-\frac{1}{RC}t}\right) u(t) \qquad t>0$$

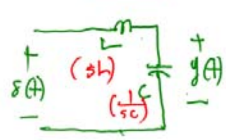
In general,



$t>0$

$$H\, x(t) = \sum_{k=-\infty}^{\infty} \lim_{\epsilon\to 0} x(k\epsilon)\, \epsilon\, H\, \frac{Rect((t-k\epsilon)|\epsilon)}{\epsilon}$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$\sin(\omega_0 t + \phi) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$



$$Y(s) = \frac{\frac{1}{sC}}{\frac{1}{sC} + sL}\, \Delta(s) = \frac{1}{s^2 LC + 1} =$$

$\Delta(s) = 1$

$$\int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) e^{-s(0)} dt = 1 \underbrace{\int_{-\infty}^{\infty} \delta(t)\, dt}_{=1} = 1 = \Delta(s)$$

$$\downarrow \mathcal{L}^{-1} \quad \frac{1}{LC}\, \frac{1}{s^2 + \left(\frac{1}{LC}\right)}$$

$$\left[\frac{1}{LC} = \left(\frac{1}{\sqrt{LC}}\right)^2 = \omega_0^2, \quad \frac{1}{\sqrt{LC}} = \omega_0\right]$$

$Y(s) = \frac{1}{LC}\frac{1}{s^2+\omega_0^2} = \frac{1}{LC}\frac{1}{(s+j\omega_0)(s-j\omega_0)}$

$= \frac{1}{LC}\left(\frac{A}{s+j\omega_0} + \frac{B}{s-j\omega_0}\right)$

$A = \cdots \frac{1}{-2j\omega_0}$

$B = \cdots \frac{1}{+2j\omega_0}$

$y(t) =$ sinusoidal function

$x(t) \rightarrow \boxed{H} \rightarrow y(t)$ LTI

$y(t)|_{x(t)=\delta(t)} = h(t)$ impulse response

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t)*h(t). \qquad (3.14)$$

Next we derive an important property of the convolution integral by making a change of variables in (3.13); let $s = (t-\tau)$. Then $\tau = (t-s)$ and $d\tau = -ds$. Equation (3.13) becomes

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{\infty}^{-\infty} x(t-s)h(s)[-ds]$$

$$= \int_{-\infty}^{\infty} x(t-s)h(s)\,ds. = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

Next we replace s with τ in the last integral, and thus the convolution can also be expressed as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau. \qquad (3.15)$$

The convolution integral is symmetrical with respect to the input signal $x(t)$ and the impulse response $h(t)$, and we have the property

$$y(t) = x(t)*h(t) = h(t)*x(t). \qquad (3.16)$$

commutative

Prob. (5th Ed) 3.8 (d)

$x(t) = e^{-t}u(t)$, $h(t) = u(t-1) - u(t-3)$

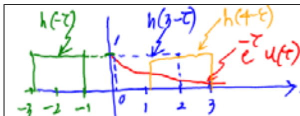
Find $y(t) = x(t) * h(t)$

Solution:

$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

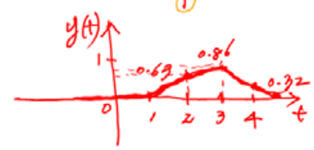
For $t \le 1$, $y(t) = 0$

For $t = 2$, $y(2) = \int_{-\infty+0}^{1} e^{-\tau}\,d\tau = -e^{-\tau}\Big|_0^1 = 1 - e^{-1} = 0.63$



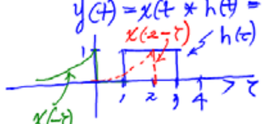
For $t=3$, $y(3) = \int_0^2 e^{-\tau}\,d\tau = -e^{-\tau}\Big|_0^2 = 1 - e^{-2} = 0.86$

For $t=4$, $y(4) = \int_1^3 e^{-\tau}\,d\tau = -e^{-\tau}\Big|_1^3 = e^{-1} - e^{-3} = 0.37 - 0.05 = 0.32$



Alternatively

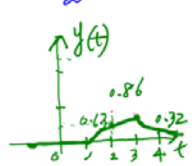
$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$



$y(t) = 0$ for $t \le 1$

$t = 2$

$y(2) = \int_1^2 e^{-(2-\tau)}\,d\tau = e^{-2}[e^{\tau}]_1^2 = e^{-2}(e^{+2} - e^{+1}) = 1 - e^{-1} = 0.63$



$y(3) = \int_1^3 e^{-(3-\tau)}\,d\tau = e^{-3}[e^{\tau}]_1^3 = e^{-3}(e^3 - e^1) = 1 - e^{-2} = 0.86$

$y(4) = \int_1^3 e^{-(4-\tau)}\,d\tau = e^{-4}[e^{\tau}]_1^3 = e^{-1} - e^{-3} = 0.37 - 0.05 = 0.32$

Properties of Convolution

1. commutative $x(t) * h(t) = h(t) * x(t)$

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \leftrightarrow h(t) \rightarrow \boxed{x(t)} \rightarrow y(t)$

2. Associative $x(t) * h_1(t) * h_2(t)$

$= [x(t) * h_1(t)] * h_2(t)$

$= x(t) * [h_1(t) * h_2(t)]$

$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t)$

$x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t)$

3. Distributive

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

Bounded Input Bounded Output (BIBO) Stability

x(t) Bounded in amplitude, i.e. $|x(t)| < M_i$ → BIBO stable system → y(t) Bounded in amplitude $|y(t)| < M_o$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Taking $|\cdot|$ (absolute value on both sides,

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \right| \le \int_{-\infty}^{\infty} |x(t-\tau)|\, |h(\tau)|\, d\tau$$

$$< M_i \underbrace{\int_{-\infty}^{\infty} |h(\tau)|\, d\tau}$$ if bounded, y(t) is bounded

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is BIBO stable if $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

sufficient condition for BIBO stability.

When system is causal. For $x(t)$ applied for $t \ge 0$, $y(t) = 0$ for $t < 0$

And $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{0}^{\infty} |h(\tau)|\, d\tau < \infty$

becomes the sufficient condition for BIBO stability

(E.g.) If $h(t) = u(t)$ (

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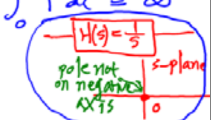
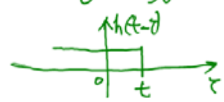
then $\int_{0}^{\infty} |h(\tau)|\, d\tau = \int_{0}^{\infty} 1\, d\tau = \infty$

not BIBO stable.



In fact when $x(t) = u(t)$,

$$y(t) = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{t} 1 \cdot u(t-\tau)\, d\tau = t$$

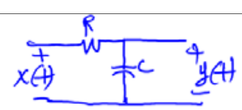


$y(t) \to \infty$ as $t \to \infty$

$|y(t)| \not< \infty$ for all t

(not BIBO stable)

(E.g.)



$$H(s) = \frac{1}{RCs + 1}$$

we learned that

for $x(t) = u(t)$, $y(t) = 1 - e^{-\frac{1}{RC}t}$

By linearity

when $x(t) = M\, u(t)$, $y(t) = M\left(1 - e^{-\frac{1}{RC}t}\right)$

$|x(t)| \le M$ for all t, $|y(t)| \le M$ for all t

BIBO stable

(Because R dissipated energy!)