ECE 103 Lecture #7 Oct 12, 2018

Differential Equation Models of LTI systems

KVL $x(t) = v_R(t) + v_C(t)$, $y(t) = v_C(t)$

$= R\,i(t) + y(t)$

$i(t) = C\dfrac{dy(t)}{dt}$ $\boxed{RC\dfrac{dy(t)}{dt} + y(t) = x(t)}$

$$RC\frac{dy(t)}{dt} + y(t) = x(t) \qquad (1)$$

$$y(t) = \underbrace{y_c(t)}_{\text{complementary solution for zero input and a particular initial condition}} + \underbrace{y_p(t)}_{\text{particular solution for a particular input \& zero initial condition}} \qquad (2)$$

$$y_c(t) = Me^{st} \qquad (3)$$

From (1) & (3), $RCMse^{st} + Me^{st} = 0$

$(RCs+1)Me^{st} = 0 \Rightarrow \boxed{s = -\frac{1}{RC}}$

$\Rightarrow y_c(t) = Me^{-\frac{1}{RC}t}$ $(3)'$

$y(t) = y_c(t) + y_p(t)$

$= Me^{-\frac{1}{RC}t} + y_p(t)$

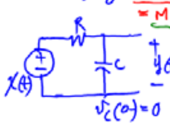
$RC\dfrac{dy_p(t)}{dt} + y_p(t) = x(t)$ (1)

For $x(t) = X u(t)$, $y_p(t) = K_1 + K_2 e^{-\frac{1}{RC}t}$ (2) is expected.

(2)→(1): $RCK_2\left(-\frac{1}{RC}\right)e^{-\frac{1}{RC}t} + \left(K_1 + K_2 e^{-\frac{1}{RC}t}\right) = X \Rightarrow K_1 = X,\ t > 0$

Thus, $y(t) = (M + K_2)e^{-\frac{1}{RC}t} + X$, $t > 0$ $\Rightarrow$ For $M = 0$, $K_2 = -X$

$= \underbrace{Me^{-\frac{1}{RC}t}}_{y_c(t)} + \underbrace{X\left(1 - e^{-t/RC}\right)}_{y_p(t)}$

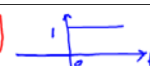


$i(t) = \delta(t)/R$

$y(t) = \frac{1}{C}\int_{0_-}^{t} i(\tau)\,d\tau$

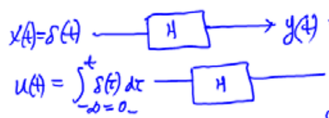
For $\underbrace{x(t) = \delta(t)}_{\text{Impulse}}$, $v_c(0_+) = \frac{1}{RC}$ and $\underbrace{y(t) = y_c(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}}_{\text{Impulse Response}}$

For $\boxed{x(t) = u(t)}$



$= \int_{-\infty}^{t}\delta(\tau)\,d\tau$

$RC\dfrac{dy(t)}{dt} + y(t) = \int_{-\infty}^{t}\delta(\tau)\,d\tau$



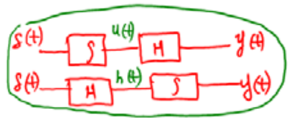
$y(t) = \frac{1}{RC}e^{-\frac{1}{RC}t} = h(t)$

$\int_{0_-}^{t} h(\tau)\,d\tau$

$= \int_{0_-}^{t}\frac{1}{RC}e^{-\frac{1}{RC}\tau}\,d\tau$

$= \frac{1}{RC}\left[-RCe^{-\frac{1}{RC}\tau}\right]_{0_-}^{t}$

$= 1 - e^{-\frac{1}{RC}t}$



Example 3.3 (5th ed)

$$\frac{dy(t)}{dt} + 2y(t) = 2$$

For $y_c(t)$, $\dfrac{dy_c(t)}{dt} + 2y_c(t) = 0$

let $y_c(t) = Me^{st}$

$sMe^{st} + 2Me^{st} = 0$

$(s+2)Me^{st} = 0 \Rightarrow \boxed{s = -2}$

$y_c(t) = Me^{-2t}$

$y(0) = M$ (initial condition)

For $y_p(t)$, $\dfrac{dy_p(t)}{dt} + 2y_p(t) = 2$, in steady state $2y_p(t) = 2 \Rightarrow \boxed{y_p(t) = 1}$

Thus, $y(t) = y(0)e^{-2t} + \leftarrow$ $y_p(t) = \overset{2}{K_1} + K_2 e^{-2t}$, $K_2 = -2$

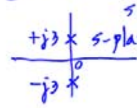
Prob. (3.29) (5th ed)

(b) Is $\dfrac{d^2y(t)}{dt^2} + 9y(t) = x(t)$ stable?

characteristic eq. for $y(t) = e^{st}$,

$(s^2 + 9)e^{st} = 0$ $s^2 + 9 = (s + j3)(s - j3) = 0$

$s = \pm j3$



not with negative real part (unstable)

Prob (3.27) Part (d)' (modified)

$$\frac{d^3y(t)}{dt^3} + \cancel{5}\,2\,\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = x(t)$$

For $y(t) = e^{st}$, $x(t) = 0$

$s^3 + 2s^2 + 4s + 3 = 0$

$\rightarrow s^3 + s^2 + s^2 + 4s + 3 = 0$

$\rightarrow s^2(s+1) + (s+1)(s+3) = 0$

$(s^2 + s + 3)(s+1) = 0$

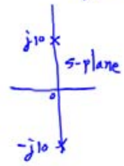
$\Rightarrow s = -1,\ s = \frac{-1 \pm \sqrt{1^2 - 4\times1\times3}}{2} = \frac{-1 \pm j\sqrt{11}}{2}$

all roots have negative real parts

(stable) ✓

Prob (3.30) $H(s) = \frac{1}{0.01s^2+1}$ $\left(= \frac{1}{LCs^2+1}\right)$, $LC = 0.01$

$$H(s) = \frac{1}{0.01s^2+1} = \frac{100}{s^2+100} = \frac{100}{(s+j10)(s-j10)}$$



$$= \frac{A}{s+j10} + \frac{B}{s-j10}$$

$$A = \frac{100}{(s+j10)(s-j10)} \times (s+j10)\Big|_{s=-j10} = \frac{100}{-j20} = +j5$$

$$B = \frac{100}{(s+j10)(s-j10)} \times (s-j10)\Big|_{s=+j10} = \frac{100}{+j20} = -j5$$

$$\Rightarrow H(s) = \frac{+j5}{s+j10} + \frac{-j5}{s-j10}$$

$$h(t) = j5\left(e^{-j10t} - e^{j10t}\right) = \underbrace{j5(-2j)}_{+10}\frac{e^{+j10t} - e^{-j10t}}{2j}$$

$= +10(\sin 10t) = \underline{10 \sin 10t}$

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

necessary & sufficient cond.

(c) For $x(t) = e^{-t}u(t)$

$$X(s) = \frac{1}{s+1}$$

$$Y(s) = H(s)X(s) = \frac{100}{s^2+100}\,\frac{1}{s+1}$$

$$= \frac{A}{s+j10} + \frac{B}{s-j10} + \frac{C}{s+1}$$

$\theta = 0^\circ - \left(-90^\circ + \tan^{-1}(-10)\right)$
$= 90^\circ - 84.29$
$= 5.71^\circ$

$$A = \frac{100}{(s+j10)(s-j10)(s+1)} \times (s+j10)\Big|_{s=-j10} = \frac{100}{(-j20)(1-j10)} = \frac{5}{\sqrt{101}}e^{+j\theta}$$

$$B = \frac{100}{(s+j10)(s-j10)(s+1)} \times (s-j10)\Big|_{s=+j10} = \frac{100}{+j20\,(1+j10)} = \frac{5}{\sqrt{101}}e^{-j\theta}$$

$$C = \frac{100}{(s+j10)(s-j10)(s+1)} \times (s+1)\Big|_{s=-1} = \frac{100}{101}$$

For $t > 0$

$$y(t) = \frac{5}{\sqrt{101}}e^{+j\theta}e^{-j10t} + \frac{5}{\sqrt{101}}e^{-j\theta}e^{+j10t} + \frac{100}{101}e^{-t} = \boxed{\frac{10}{\sqrt{101}}\cos(10t-\theta) + \frac{100}{101}e^{-t}}$$

$x(t)$ — H — $y(t)$

$h(t) = 10 \sin 10t$

when $x(t) = \sin 10t$ $\quad |x(t)| \le 1$ for all t (bounded)

$$Y(s) = X(s)\,H(s) = \frac{10}{s^2+100}\;\frac{100}{s^2+100}$$

$$= 10\left(\frac{10}{s^2+10^2}\right)^2 \xrightarrow{\mathcal{L}^{-1}} \frac{\cancel{1}}{10}\left(\sin 10t - 10t\cos 10t\right)$$

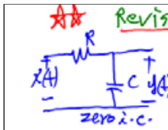
$= y(t)$

$|y(t)| \not< \infty$ as $t \to \infty$

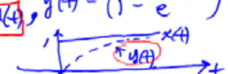
TABLE 3.2 Key Equations of Chapter 3

Equation Title	Equation Number	Equation
Unit impulse response	(3.9)	$\delta(t) \rightarrow h(t)$
Convolution integral	(3.15)	$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$
Convolution with a unit impulse	(3.18)	$\delta(t)*g(t) = g(t)$
Convolution integral of an inverse system	(3.32)	$x(t)*h(t)*h_i(t) = x(t)$
Convolution integral of a causal system	(3.37)	$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau$
Condition on impulse response for BIBO stability	(3.39)	$\int_{-\infty}^{\infty} \lvert h(t)\rvert\, dt \le \infty$
Derivation of step response from impulse response	(3.41)	$s(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau = \int_{0}^{\infty} h(t-\tau)\,d\tau$
Derivation of impulse response from step response	(3.43)	$h(t) = \frac{ds(t)}{dt}$
Linear differential equation with constant coefficients	(3.50)	$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}$
Characteristic equation	(3.55)	$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$
Solution of homogeneous equation	(3.57)	$y_c(t) = C_1e^{s_1t} + C_2e^{s_2t} + \cdots + C_ne^{s_nt}$
Transfer function	(3.70)	$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$
Steady-state response to a complex exponential input	(3.71)	$x(t) = Xe^{s_1t} \rightarrow y_{ss}(t) = XH(s_1)e^{s_1t}$
Steady-state response to a sinusoidal input	(3.73)	$\lvert X\rvert \cos(\omega_1 t + \phi) \rightarrow \lvert X\rvert\lvert H(j\omega_1)\rvert \cos[\omega_1 t + \phi + \angle H(j\omega_1)]$
Input expressed as sum of complex exponentials	(3.74)	$x(t) = \sum_{k=1}^{N} X_k e^{s_kt}$
Output expressed as sum of complex exponentials	(3.75)	$y_{ss}(t) = \sum_{k=1}^{N} X_k H(s_k)e^{s_kt}$
Transfer function expressed as integral of impulse response	(3.77)	$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$

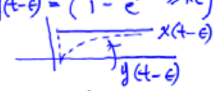
★★ Revisit of Impulse Response ($\delta(t)$), Step input Response ($u(t)$)



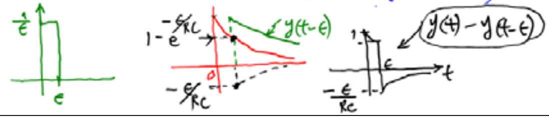
For $\boxed{x(t) = u(t)}$, $y(t) = \left(1 - e^{-t/RC}\right)u(t)$

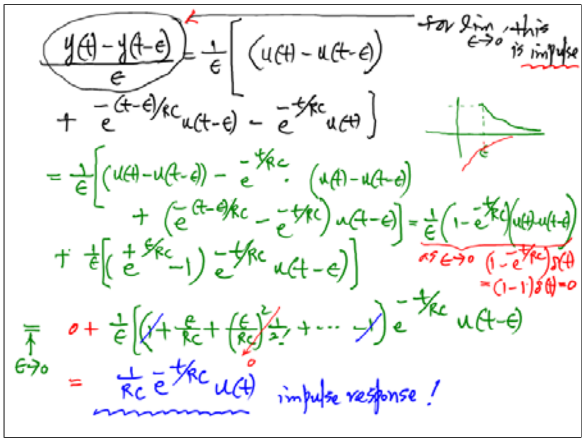


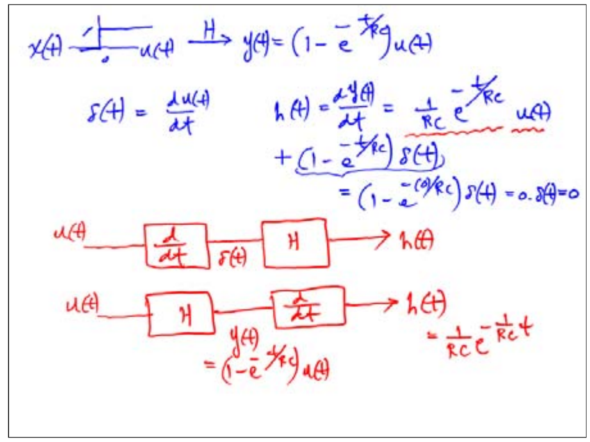
For $\boxed{x(t-\epsilon) = u(t-\epsilon)}$ $\rightarrow$ $y(t-\epsilon) = \left(1 - e^{-(t-\epsilon)/RC}\right)u(t-\epsilon)$

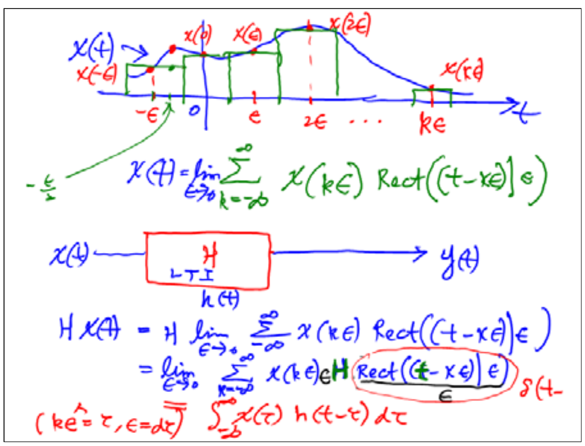


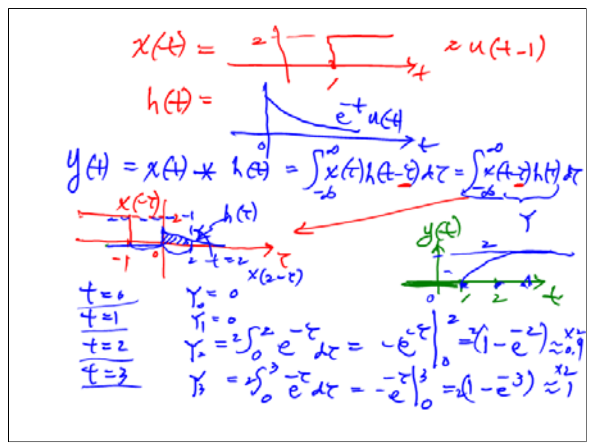
$x(t) - x(t-\epsilon)$ $\rightarrow$ $\left(1 - e^{-t/RC}\right)u(t) - \left(1 - e^{-(t-\epsilon)/RC}\right)u(t-\epsilon)$




for lim ε→0, this is impulse
impulse response !


H
d/dt
H
h(t)
h(t)


H
LTI
h(t)
y(t)


t=0
t=1
t=2
t=3