ECE 103 Lecture 9, Oct. 17, 2018

For periodic signals with period T_0

$$\omega_0 = 2\pi f_0 = 2\pi\left(\frac{1}{T_0}\right),$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t}$$

$$= C_0 + \sum_{k=1}^{\infty}\left(C_{-k} e^{j(-k)\omega_0 t} + C_k e^{jk\omega_0 t}\right)$$

$$= C_0 + \sum_{k=1}^{\infty}\left(C_{-k} e^{-jk\omega_0 t} + C_k e^{jk\omega_0 t}\right)$$

Chapter 4 Fourier Series

TABLE 4.2 Forms of the Fourier Series

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t};\quad C_k = \lvert C_k\rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k\rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty}(A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$ $2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\omega_0 t} = \underbrace{C_0 e^{j(0)\omega_0 t}}_{=C_0} + \sum_{k=1}^{\infty}\left(C_{-k} e^{-jk\omega_0 t} + C_k e^{jk\omega_0 t}\right) \quad (4)$$

where $$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt \quad (5)$$

$$C_{-k} = \frac{1}{T_0}\int_{T_0} x(t) e^{-j(-k)\omega_0 t} dt \quad (6)$$

complex conjugate

$$C_k^* = \frac{1}{T_0}\int_{T_0} x(t) e^{+jk\omega_0 t} = C_{-k} \quad (7)$$

$$|C_k| = |C_{-k}| \quad (8)$$

$$\theta_{-k} = -\theta_k \quad (9)$$

see next page

$$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t)(\cos k\omega_0 t - j \sin k\omega_0 t) dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t) \cos k\omega_0 t\, dt \quad \text{(real part)}$$

$$-j\frac{1}{T_0}\int_{T_0} x(t) \sin k\omega_0 t\, dt \quad \text{(imaginary part)}$$

$$= A_k - jB_k \qquad A_k = A_{-k},\quad B_k = -B_{-k}$$

$$|C_k| = \sqrt{A_k^2 + B_k^2} = |C_{-k}|$$

$$\angle C_k = \tan^{-1}\frac{(-B_k)}{A_k} \text{ and } \angle C_{-k} = \tan^{-1}\frac{(-B_{-k})}{A_{-k}} = \tan^{-1}\frac{B_k}{A_k} = -\angle C_k$$

$$\Rightarrow C_{-k} = |C_{-k}| e^{+j\theta_{-k}}$$

$$\underset{(8),(9)}{=} |C_k| e^{-j\theta_k} = (C_k)^* \quad (10)$$

thus from (4) & (10)

$$x(t) = C_0 + \sum_{k=1}^{\infty} |C_k|\left[e^{-j(k\omega_0 t + \theta_k)} + e^{+j(k\omega_0 t + \theta_k)}\right]$$

$$\left(e^{j\phi} + e^{-j\phi} = 2\cos\phi\right)$$

$$= C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k) \quad (11)$$

From (11) $$x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k)$$

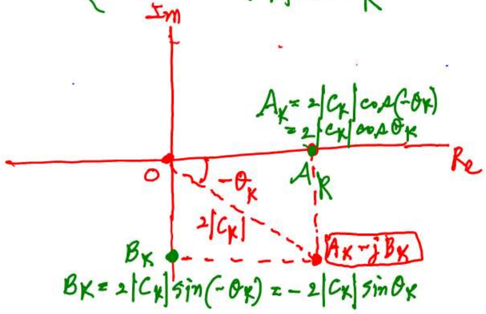
$$= C_0 + \sum_{k=1}^{\infty}\left[2|C_k| \cos k\omega_0 t \cdot \cos\theta_k - 2|C_k| \sin k\omega_0 t \cdot \sin\theta_k\right]$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= C_0 + \sum_{k=1}^{\infty}\left[\underline{2|C_k|\cos\theta_k} \cdot \cos k\omega_0 t + \underline{(-2|C_k|\sin\theta_k)} \cdot \sin k\omega_0 t\right]$$

$$= C_0 + \sum_{k=1}^{\infty}\left(A_k \cos k\omega_0 t + B_k \sin k\omega_0 t\right) \quad (12)$$

where $\begin{cases} A_k = 2|C_k| \cos\theta_k \\ B_k = -2|C_k| \sin\theta_k \end{cases}$



Calculation of C_n

$$C_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{T_0}\left(\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}\right) e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{T_0}\sum_{k=-\infty}^{\infty} C_k e^{j(k-n)\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{T_0} C_n e^{\overset{=0}{j(n-n)\omega_0 t}}\,dt \quad (k=n)$$

$$+ \frac{1}{T_0}\int_{T_0}\sum_{k=-\infty}^{\infty} C_k e^{j(k-n)\omega_0 t}\,dt \quad (k\neq n)$$

$$= \frac{1}{T_0}\int_{T_0} C_n e^0\,dt + \underbrace{\sum_{\substack{n=-\infty \\ n\neq k}}^{\infty}\int_{T_0} C_k \overbrace{e^{+j(k-n)\omega_0 t}}^{\text{periodic}}\,dt}_{=0}$$

$$= \cancel{\frac{1}{T_0}} C_n \cancel{T_0} + 0 = C_n$$

$$\boxed{C_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\,dt}$$

Example 1 For $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_0) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$

$$C_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\,dt$$

$\omega_0 = 2\pi/T_0$

$$= \frac{1}{T_0}\int_{T_0} \delta(t-nT_0)\, e^{-jn\omega_0 t}\,dt \quad \binom{k=n}{\text{term}}$$

$$+ \frac{1}{T_0}\int_{T_0}\sum_{\substack{k=-\infty \\ k\neq n}}^{\infty} \delta(t-kT_0)\, e^{-jn\omega_0 t}\,dt$$

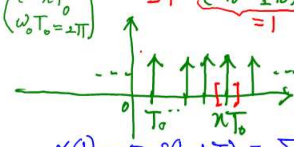
$e^{-j2\pi n^2}$

$$= \frac{1}{T_0}\underbrace{e^{-jn\omega_0(nT_0)}}_{=1}\underbrace{\int_{(nT_0-\frac{1}{2}T_0)}^{(nT_0+\frac{1}{2}T_0)}\delta(t-nT_0)\,dt}_{=1} + \underbrace{\frac{1}{T_0}\int_{(nT_0-\frac{1}{2}T_0)}^{(nT_0+\frac{1}{2}T_0)}\sum \cdots dt}$$ $= \frac{1}{T_0}$

$\begin{pmatrix} t = nT_0 \\ \omega_0 T_0 = 2\pi \end{pmatrix}$

For $[nT_0 - \frac{1}{2}T_0 \quad nT_0 + \frac{1}{2}T_0]$
$\delta(t-kT) = 0$ for $k\neq 0$
Thus this term is 0

$= \frac{1}{T_0} = C_n$



$$x(t) = \sum \delta(t-kT_0) = \sum \frac{1}{T_0} e^{jk\omega_0 t}$$

Recap

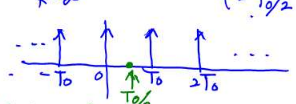
$$\sum_{k=-\infty}^{\infty} \delta(t-kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \qquad (1)$$

LHS of (1)
When $t = T_0$, $\sum_{k=-\infty}^{\infty} \delta(T_0 - kT_0)$
$= 0 + \cdots + 0 + \delta(0) + 0 + \cdots$ $= \infty$
$\uparrow k=1$

RHS of (1)
At $t = T_0$, $\frac{1}{T}\sum_{k=-\infty}^{\infty} \underbrace{e^{j\overbrace{k\omega_0 T_0}^{=2\pi}}}_{=1}$
$= \infty$

Let us also check at $t = \frac{T_0}{2}$

$$\sum_{k=-\infty}^{\infty} \delta(t-kT_0) \overset{\uparrow}{\underset{t=T_0/2}{=}} 0$$

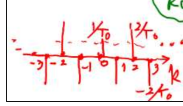


At $t = T_0/2$

$$\frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \overset{\uparrow}{=} \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\pi}$$

$t = T_0/2$
$k\omega_0 T_0/2 = k\pi$

$$= \frac{1}{T_0}\left(\sum_{k=-\infty}^{\infty}\left(\cos k\pi + j\overset{=0}{\sin k\pi}\right)\right)$$

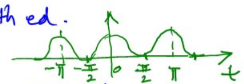


$$= \frac{1}{T_0}\left(\sum_{k=-\infty}^{\infty}\cos k\pi\right) = \frac{1}{T_0} + \sum_{k=1}^{\infty}\frac{2}{T_0}\cos k\pi$$

$= ?$

(e.g) Prob. 4.2 (ii) 5th ed.

$x(t) = \cos^2 t$



$$x(t) = \cos^2 t = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

the period of $\cos t = 2\pi$

" $\cos^2 t = \pi \Rightarrow \omega_0 = 2\pi \frac{1}{\pi} = 2$

(ii) $x(t) = \cos^2(t) = \frac{1}{2}[1 + \cos(2t)]$

(a) Exponential form: $\omega_0 = 2$

$$= \sum C_k e^{jk\omega_0 t} = C_0 + C_{-1} e^{-j2t} + C_1 e^{+j2t}$$

$$\frac{1}{2}[1 + \cos(2t)] = \frac{1}{2} + \frac{1}{4}e^{j2t} + \frac{1}{4}e^{-j2t}$$

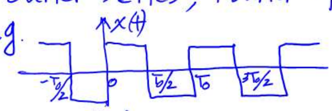
$C_0 = \frac{1}{2}$

$C_1 = \frac{1}{4}, C_{-1} = \frac{1}{4}$

$C_k = 0, k \neq 0, 1, -1$

Fourier Series, Fourier Spectrum

e.g.



$\begin{cases} \text{period} = T_0 \\ f_0 = 1/T_0 \end{cases}$

$$x(t) = \begin{cases} V, & 0 < t < T_0/2 \\ -V, & T_0/2 < t < T_0 \end{cases}$$

$x(t) = \sum C_k e^{jk\omega_0 t}$, where $\omega_0 = 2\pi \frac{1}{T_0}$

$$C_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} V e^{-jk\omega_0 t}\, dt + \int_{T_0/2}^{T_0} -V e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{V}{T_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\Big|_0^{T_0/2} - \frac{e^{-jk\omega_0 t}}{-jk\omega_0}\Big|_{T_0/2}^{T_0}\right]$$

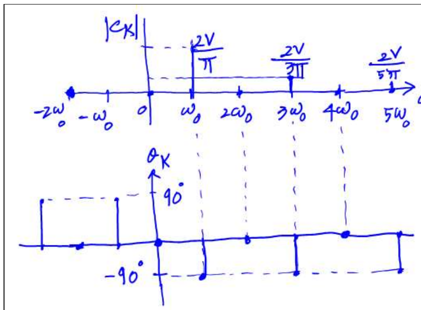
$$= \frac{V}{T_0}\left[\frac{e^{-jk\omega_0 T_0/2} - 1}{-jk\omega_0} - \frac{e^{-jk\omega_0 T_0} - e^{-jk\omega_0 T_0/2}}{-jk\omega_0}\right]$$

$$= \frac{V}{T_0}\frac{1}{-jk\omega_0}\left[e^{-jk\pi} - 1 - \underbrace{e^{-jk2\pi}}_{1} + e^{-jk\pi}\right]$$

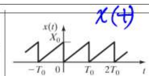
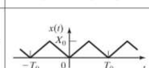
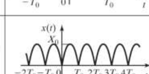
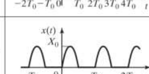
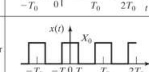
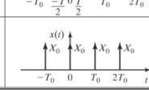
$$= \frac{V}{-jk\underbrace{\omega_0 T_0}_{=2\pi}}\left[2e^{-jk\pi} - 2\right]$$

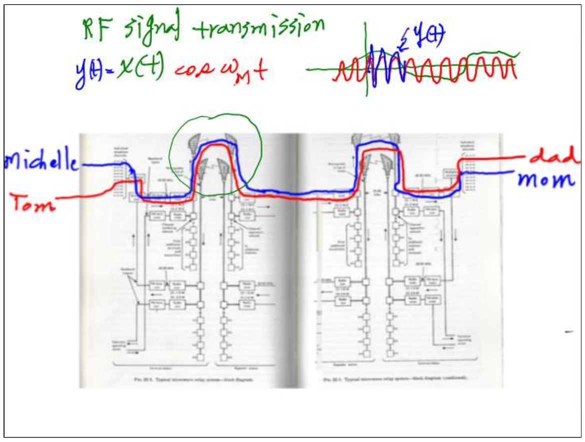
$$= \frac{V}{-jk\pi}\left[e^{-jk\pi} - 1\right]$$

$$C_k = \begin{cases} 0, & k \text{ even} \\ \frac{2V}{+jk\pi}, & k \text{ odd} \end{cases} \quad \left|\frac{2V}{jk\pi}\right| = \frac{2V}{k\pi}, \ \theta = -90^\circ$$



Frequency Spectrum of the square wave $x(t) = \begin{cases} V, & 0 < t < T_0/2 \\ -V, & T_0/2 < t < T_0 \end{cases}$

	$x(t)$	C_0	$C_k, k \neq 0$	
2. Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2 - 1)}$	
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2 - 1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\operatorname{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	



$$y(t) = x(t)\, M_2 \cos \omega_M t$$

$$= M_1 \cos \omega_0 t \cdot M_2 \cos \omega_M t$$

(when $x(t) = M_1 \cos \omega_0 t$, $\omega_M \gg \omega_0$)

$$= M_1 M_2 \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot \frac{e^{j\omega_M t} + e^{-j\omega_M t}}{2}$$

$$= \frac{M_1 M_2}{4}\left(e^{j(\omega_0+\omega_M)t} + e^{+j(\omega_M-\omega_0)t} + e^{-j(\omega_0+\omega_M)t} + e^{-j(\omega_M-\omega_0)t}\right)$$