

Motivational Examples

1. signal multiplexing $x(t) \times c(t)$

2. EC of signal periodic

3. Butterworth Filter

Lab # 6

Amplitude Time

Signal Transformation

Even & Odd Components of signals

(i) odd signal $x_c(t) = -x_c(t)$
or $x_c(t) = -x_c(-t)$

e.g. $x(t) = \sin t$ is odd
 $x_c(t) = \sin t$ is odd
 $x_e(t) = \cos t$ is even

How to obtain $x_e(t)$, $x_o(t)$ & $x(t)$?

$x(t) = x_e(t) + x_o(t)$
 $x(-t) = x_e(-t) + x_o(-t)$
 $x_e(t) = \frac{x(t) + x(-t)}{2}$
 $x_o(t) = \frac{x(t) - x(-t)}{2}$

Representation of signals

Mathematical Expression of signals

e.g. $x(t) = \text{Rect}(t/T)$

$x(t) = \text{Rect}(t/T) = \begin{cases} 1 & -T/2 \leq t \leq T/2 \\ 0 & \text{elsewhere} \end{cases}$

$x(t) = \text{Rect}(t/T) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi t/T)}{n}$

Periodic signals

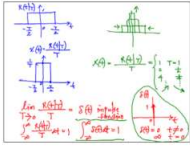
if $x(t) = x(t+T)$ for $T > 0$, then $x(t)$ is periodic with period T . The minimum value of T for which $x(t) = x(t+T)$ is called the fundamental period T_0 .

Periodic signals $x(t) = x(t+T)$ for all t , $T > 0$

e.g. $x(t) = A \cos(\omega t)$, where $\omega = 2\pi/T$, $T = \text{period}$

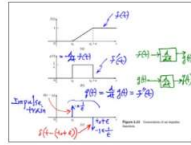
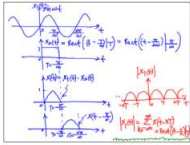
$x(t) = A \cos(\omega t) = A \cos(2\pi t/T)$

$x(t) = A \cos(2\pi t/T) = A \cos(2\pi(t+T)/T) = A \cos(2\pi t/T + 2\pi) = A \cos(2\pi t/T)$

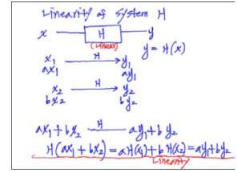


2. Impulse $\delta(t) = \lim_{\epsilon \rightarrow 0} \frac{\text{rect}(t/\epsilon)}{\epsilon}$

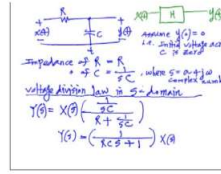
Time Derivatives of signals



| Property | Equation |
|---------------------------------------|---|
| Linearity | $H(ax+by) = aH(x) + bH(y)$ |
| Time Invariance | $H(x(t-t_0)) = y(t-t_0)$ |
| Stability | $\int_{-\infty}^{\infty} y(t) dt < \infty$ |
| Memoryless | $y(t) = 0$ for $x(t) = 0$ |
| Time Reversal | $H(x(-t)) = y(-t)$ |
| Time Shifting | $H(x(t-t_0)) = y(t-t_0)$ |
| Time Scaling | $H(x(at)) = \frac{1}{ a } y(t/a)$ |
| Time Folding | $H(x(-t)) = y(-t)$ |
| Time Inversion | $H(x(-t)) = y(-t)$ |
| Time Shifting and Scaling | $H(x(at-t_0)) = \frac{1}{ a } y((t-t_0)/a)$ |
| Time Shifting and Folding | $H(x(-t-t_0)) = y(-(t-t_0))$ |
| Time Shifting and Scaling and Folding | $H(x(-at-t_0)) = \frac{1}{ a } y(-(t-t_0)/a)$ |



Linear Time Invariant (LTI) systems



$x(t) \rightarrow [H(s)] \rightarrow y(t)$

For $ax_1(t) + bx_2(t)$

$Y(s) = \left(\frac{1}{RCs + 1} \right) (aX_1(s) + bX_2(s))$
 $= a \left(\frac{1}{RCs + 1} \right) X_1(s) + b \left(\frac{1}{RCs + 1} \right) X_2(s)$
 $= aY_1(s) + bY_2(s)$

Linear ✓

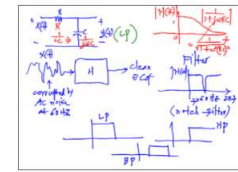
mobile phone battery RC circuit

capacitor $2100 \mu F$
 $= 2100 \times 10^{-6} F$
 $= 2.1 \times 10^{-3} F$

Terminal voltage = 3.2V
 $C = \frac{Q}{V} = \frac{1.44 \text{ Ah}}{3.2 \text{ V}} = 0.45 \text{ Ah/V}$

if battery charging time is 90 minutes
 in 900 seconds = 5RC (5 time constants)
 $5 \times 2.1 \times 10^{-3} \times 3.2 = 0.336 \text{ Ah}$

if $R = 1 \Omega$, it will take approx 0.336 hr



Mobile phone charger Example

$H(s) = \frac{1}{1 + RCs}$ is magnitude / phase angle

$= \frac{1}{\sqrt{1 + (RC\omega)^2}} \angle -\tan^{-1} RC\omega$

at $\omega = 0$
 $H(0) = \frac{1}{1 + 0} = 1 \angle -\tan^{-1} 0$

at $\omega = \frac{1}{RC}$
 $H\left(\frac{1}{RC}\right) = \frac{1}{\sqrt{1 + 1}} = \frac{1}{\sqrt{2}} \angle -\tan^{-1} 1$

at $\omega = \frac{1}{RC}$
 $H\left(\frac{1}{RC}\right) = \frac{1}{\sqrt{1 + 1}} = \frac{1}{\sqrt{2}} \angle -\tan^{-1} 1$

Convolution

$y(t) = x(t) * h(t)$
 $= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

$\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = h(t) * x(t)$ (Commutative)

$x(t) = u(t)$, $h(t) = e^{-t} u(t)$
 $x(t) * h(t) = \int_{-\infty}^{\infty} u(\tau) e^{-(t-\tau)} u(t-\tau) d\tau$
 $= \int_0^t e^{-t+\tau} d\tau = e^{-t} (e^t - 1) = 1 - e^{-t}$

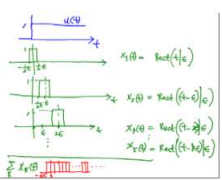
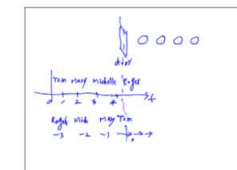
$h(t) * x(t) = \int_{-\infty}^{\infty} u(t-\tau) e^{-\tau} u(\tau) d\tau$
 $= \int_0^t e^{-\tau} d\tau = 1 - e^{-t}$

$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$

Convolution

Convolution (def by Webster dictionary)
 A uniting, joining, or welding together

any of the irregular folds or wrinkles on the surface of the brain (with the skull)



$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

$x(t) = \text{rect}(t/c)$
 $h(t) = \text{rect}(t/c)$

$y(t) = \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$

$= \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$

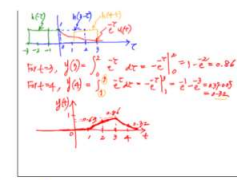
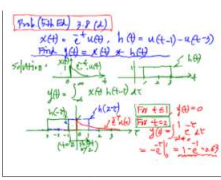
$= \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$

$x(t) * h(t)$

if we set $\tau = t - \tau$
 $\rightarrow \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$
 $= \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$

$x(t) = \text{rect}(t/c)$
 $h(t) = \text{rect}(t/c)$

$y(t) = \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$
 $= \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$



Example of Convolution

Alternative way

$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$

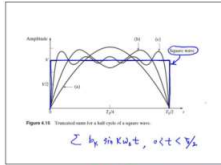
$x(t) = \text{rect}(t/c)$
 $h(t) = \text{rect}(t/c)$

$y(t) = \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$

$= \int_{-\infty}^{\infty} \text{rect}(\tau/c) \text{rect}(t-\tau/c) d\tau$

Properties of Convolution

- Commutative $x(t) * h(t) = h(t) * x(t)$
- Associative $(x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t))$
- Distributive $x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$



Chapter 4 Fourier Series

TABLE 4.2 Forms of the Fourier Series

| Name | Equation |
|---------------------------|---|
| Exponential | $\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$, $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$ |
| Combined exponential form | $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$ |
| Trigonometric | $A_k \cos(k\omega_0 t + \theta_k)$, $B_k \sin(k\omega_0 t + \phi_k)$ |
| Exponential | $A_k = \frac{1}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$ |
| Trigonometric | $B_k = \frac{1}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$ |
| Complex coefficient | $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$ |

Conditions:
 Periodic: $x(t) = x(t + nT)$
 Average value: $A_{avg} = \frac{1}{T} \int_0^T x(t) dt$
 Average value: $A_{avg} = \frac{1}{T} \int_0^T x(t) dt$
 Average value: $A_{avg} = \frac{1}{T} \int_0^T x(t) dt$

Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \quad (1)$$

C_k is a complex number
 $C_k = |C_k| e^{j\theta_k} \quad (2)$

Form (1) & (2):
 $x(t) = \sum_{k=-\infty}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)} \quad (3)$

$$= |C_0| e^{j0} + \sum_{k=1}^{\infty} (|C_k| e^{-j\theta_k} e^{jk\omega_0 t} + |C_k| e^{j\theta_k} e^{-jk\omega_0 t})$$

where $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$
 complex coefficient $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$
 $|C_k| = |C_{-k}|$
 $\theta_k = -\theta_{-k}$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} = C_0 + \sum_{k=1}^{\infty} (C_k e^{jk\omega_0 t} + C_{-k} e^{-jk\omega_0 t})$$

where $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$
 complex coefficient $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$
 $|C_k| = |C_{-k}|$
 $\theta_k = -\theta_{-k}$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T x(t) (\cos(k\omega_0 t) - j \sin(k\omega_0 t)) dt$$

$$= \frac{1}{T} \int_0^T x(t) \cos(k\omega_0 t) dt - j \frac{1}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

Real part: $A_k = \frac{1}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$
 Imaginary part: $B_k = -\frac{1}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$
 $C_k = A_k - jB_k$
 $|C_k| = \sqrt{A_k^2 + B_k^2}$
 $\theta_k = \tan^{-1} \left(\frac{-B_k}{A_k} \right)$

How to find coefficient C_k ?

$$\Rightarrow C_k = |C_k| e^{j\theta_k}$$

$$\sum_{k=-\infty}^{\infty} |C_k| e^{j\theta_k} e^{jk\omega_0 t} = (C_k)^*$$

where $(C_k)^* = C_{-k}$
 $\theta_k = -\theta_{-k}$
 $x(t) = C_0 + \sum_{k=1}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)} + \sum_{k=1}^{\infty} |C_k| e^{-j(k\omega_0 t + \theta_k)}$

From (1) $x(t) = C_0 + \sum_{k=1}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)}$
 $= C_0 + \sum_{k=1}^{\infty} |C_k| \cos(k\omega_0 t + \theta_k)$
 $= C_0 + \sum_{k=1}^{\infty} |C_k| \cos(k\omega_0 t + \theta_k)$
 $= C_0 + \sum_{k=1}^{\infty} |C_k| \cos(k\omega_0 t + \theta_k)$
 $= C_0 + \sum_{k=1}^{\infty} |C_k| \cos(k\omega_0 t + \theta_k)$

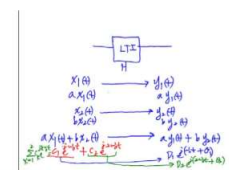
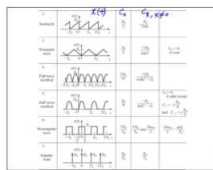
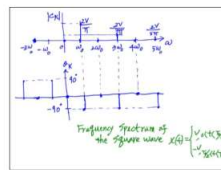
Fourier Series, Fourier Spectrum
 e.g. $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$
 $x(t) = \sum_{k=-\infty}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)}$
 $x(t) = \sum_{k=-\infty}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)}$

$$= \sum_{k=-\infty}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)}$$

$$= \sum_{k=-\infty}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)}$$

$$= \sum_{k=-\infty}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)}$$

$$C_k = |C_k| \angle C_k = \theta_k = |C_k| e^{j\theta_k}$$



$$x(t) \rightarrow H \rightarrow y(t)$$

$$X(\omega) \rightarrow H(\omega) \rightarrow Y(\omega)$$

$$y(t) = x(t) * h(t) \rightarrow Y(\omega) = X(\omega) H(\omega)$$

$$Y(\omega) = H(\omega) X(\omega)$$

$$= H(\omega) \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} C_k H(k\omega_0) e^{jk\omega_0 t}$$

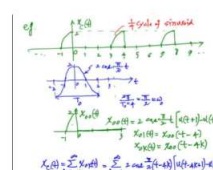
$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} |C_k| e^{j(k\omega_0 t + \theta_k)}$$

$$= \sum_{k=-\infty}^{\infty} |C_k| \cos(k\omega_0 t + \theta_k)$$

TABLE 4.3 Real Expansion of C_k

| Expansion Name | Equation |
|---------------------------|---|
| Exponential | $\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$, $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$ |
| Combined exponential form | $C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$ |
| Trigonometric | $A_k \cos(k\omega_0 t + \theta_k)$, $B_k \sin(k\omega_0 t + \phi_k)$ |
| Exponential | $A_k = \frac{1}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$ |
| Trigonometric | $B_k = \frac{1}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$ |



$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T x(t) (\cos(k\omega_0 t) - j \sin(k\omega_0 t)) dt$$

$$= \frac{1}{T} \int_0^T x(t) \cos(k\omega_0 t) dt - j \frac{1}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

$$= \frac{1}{T} \int_0^T x(t) \cos(k\omega_0 t) dt - j \frac{1}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

$$= \frac{1}{T} \int_0^T x(t) \cos(k\omega_0 t) dt - j \frac{1}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

Fourier Transformation
 $\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = X(\omega)$

$$\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} x(t) \sin(\omega t) dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} x(t) \sin(\omega t) dt$$

Handwritten notes on Fourier transforms and Bode plots. Includes the definition of the Fourier transform:

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Examples of functions and their transforms are shown, along with Bode magnitude and phase plots for various transfer functions.

2

Handwritten notes on Bode plots. A central box contains the text "Bode Plot".

Examples of Bode magnitude and phase plots are shown for different transfer functions, including asymptotic approximations.

Handwritten notes on the magnitude response of a filter. Includes an example of a transfer function:

$$H(\omega) = \frac{1}{(1 + j\omega)^2}$$

The magnitude response is plotted on a Bode plot, showing a roll-off rate of -40 dB/decade. The phase response is also shown, starting at 0 degrees and approaching -180 degrees.

Handwritten notes on Shannon's Sampling Theorem. A central box contains the text "Shannon's Sampling Theorem".

The theorem states that a signal with a maximum frequency f_m can be perfectly reconstructed from its samples if the sampling rate f_s is greater than $2f_m$.

Examples of sampling a cosine wave and the resulting spectrum are shown.

Handwritten notes on aliasing error and signal reconstruction. A central box contains the text "Aliasing error" and "signal reconstruction".

The notes explain how aliasing occurs when the sampling rate is too low, and how a low-pass filter (LP-filter) can be used to reconstruct the original signal from its samples.

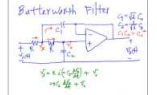
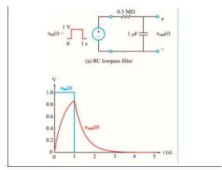
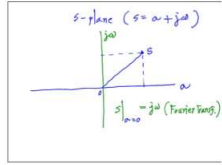
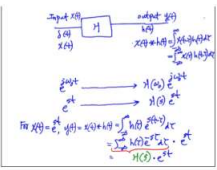
Handwritten notes on additional eText used, including a diagram of a feedback control system and a circuit diagram of an RLC network.

The feedback control system diagram shows the flow from reference to error, through a controller and plant, to the output, which is then fed back to the error signal.

The RLC network diagram shows a circuit with a resistor, inductor, and capacitor, and its corresponding transfer function.

| Time Domain | Frequency Domain | Complex Domain | Fourier Transform | Inverse Transform |
|--------------------|---|------------------------|--|--|
| $f(t)$ | $F(\omega)$ | $s = \sigma + j\omega$ | $\int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$ | $\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$ |
| $\delta(t)$ | 1 | s | $\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} dt = 1$ | $\int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega = 2\pi \delta(t)$ |
| $e^{j\omega_0 t}$ | $2\pi \delta(\omega - \omega_0)$ | $s = j\omega_0$ | $\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt = 2\pi \delta(\omega - \omega_0)$ | $\int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$ |
| $\cos(\omega_0 t)$ | $\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $s = \pm j\omega_0$ | $\int_{-\infty}^{\infty} \cos(\omega_0 t) e^{-j\omega t} dt = \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $\int_{-\infty}^{\infty} \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] e^{j\omega t} d\omega = \cos(\omega_0 t)$ |
| $\sin(\omega_0 t)$ | $\pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $s = \pm j\omega_0$ | $\int_{-\infty}^{\infty} \sin(\omega_0 t) e^{-j\omega t} dt = \pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $\int_{-\infty}^{\infty} \pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] e^{j\omega t} d\omega = \sin(\omega_0 t)$ |

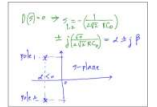
$t \quad \omega \quad s$



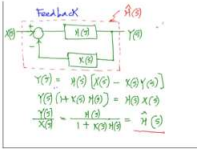
A Butterworth Filter Example

$$\begin{aligned}
 |H(j\omega)|^2 &= \frac{1}{1 + \omega^{2n}} \\
 &= \frac{1}{1 + \omega^{2n}} = \frac{1}{1 + \omega^{2n}}
 \end{aligned}$$

$$\begin{aligned}
 H(s) &= \frac{1}{\prod_{k=1}^n (s - s_k)} \\
 &= \frac{1}{\prod_{k=1}^n (s - j\omega_k)}
 \end{aligned}$$



$$\begin{aligned}
 |H(j\omega)|^2 &= \frac{1}{1 + \omega^{2n}} \\
 &= \frac{1}{1 + \omega^{2n}}
 \end{aligned}$$

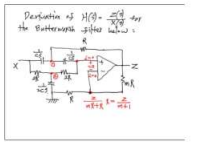
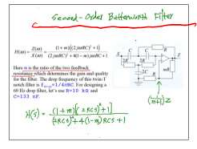
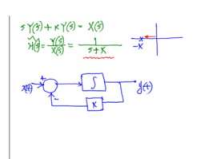
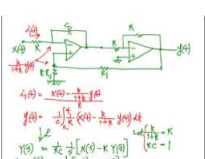
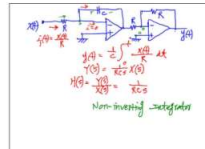
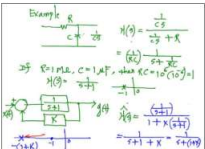


Feedback

$$\begin{aligned}
 Y(s) &= X(s) \frac{K}{1 + KX(s)} \\
 \frac{Y(s)}{X(s)} &= \frac{K}{1 + KX(s)}
 \end{aligned}$$

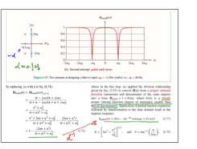
$$\begin{aligned}
 X(s) &= \frac{1}{s} \\
 Y(s) &= \frac{K}{1 + KX(s)} \\
 &= \frac{K}{1 + \frac{K}{s}} = \frac{Ks}{s + K}
 \end{aligned}$$

$$\begin{aligned}
 \frac{Y(s)}{X(s)} &= \frac{Ks}{s + K} \\
 &= \frac{K(s + K) - K^2}{s + K} = K - \frac{K^2}{s + K}
 \end{aligned}$$

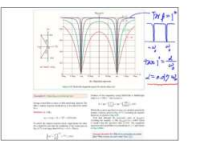


$$\begin{aligned}
 H(s) &= \frac{1}{s^2 + \sqrt{2}s + 1} \\
 &= \frac{1}{(s + j\frac{\sqrt{2}}{2})(s - j\frac{\sqrt{2}}{2}) + 1}
 \end{aligned}$$

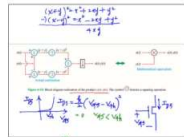
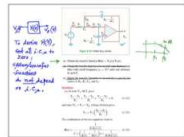
$$\begin{aligned}
 H(s) &= \frac{1}{s^2 + \sqrt{2}s + 1} \\
 &= \frac{1}{(s + j\frac{\sqrt{2}}{2})(s - j\frac{\sqrt{2}}{2}) + 1}
 \end{aligned}$$



$$\begin{aligned}
 |H(j\omega)|^2 &= \frac{1}{1 + \omega^4} \\
 &= \frac{1}{1 + \omega^4}
 \end{aligned}$$



$$\begin{aligned}
 |H(j\omega)|^2 &= \frac{1}{1 + \omega^4} \\
 &= \frac{1}{1 + \omega^4}
 \end{aligned}$$



$H(s)$ derivation

How to compute xy ?

A hardware implementation

