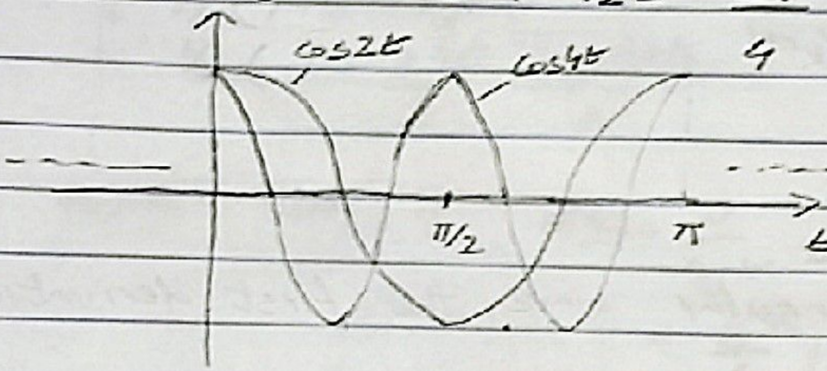
EE 103 HW# 3

[4.1] $f(t) = \cos 2t + 3\cos 4t$,

(a) To determine period of $f(t)$

① Period of $\cos 2t$ is : $T_1 = \frac{2\pi}{2} = \pi$.

② Period of $\cos 4t$ is : $T_2 = \frac{2\pi}{4} = \pi/2$.



∴ Fundamental period of $f(t) = \pi$

The Fourier Series expansion is :

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$$

$$= \sum_{k=0}^{\infty} C_k e^{jk(2)t}$$

$$= C_0 + C_1 e^{2jt} + C_2 e^{4jt} + C_3 e^{6jt} + C_4 e^{8jt} + \ldots\ldots \quad ①$$

Expand $f(t)$ is exponential form.

$$f(t) = \frac{e^{2t} + e^{-2t}}{2} + 3\frac{(e^{4t} + e^{-4t})}{2}$$

$$= \frac{3}{2}e^{-4t} + \frac{1}{2}e^{-2t} + \frac{1}{2}e^{2t} + \frac{3}{2}e^{4t} \quad ②$$

Compare ① & ②

ⓐ $C_0 = 0$

ⓑ $C_1 = 1/2$

ⓒ $C_2 = 3/2$

ⓓ $C_3 = 0,\ C_4 = 0, \dots$ $\therefore$ for $k \geq 3$ $C_k = 0$

[4.4] $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$ is periodic

show $x(t-t_0)$ is given by $\hat{x}(t) = \sum_{k=-\infty}^{\infty} \hat{C}_k e^{jk\omega_0 t}$

where $|\hat{C}_k| = |C_k|$ & $\angle \hat{C}_k = \angle C_k - k\omega_0 t_0$

Replace $x(t)$ by $x(t-t_0)$ in $\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0(t-t_0)}$$

$$= \sum_{k=-\infty}^{\infty} \left[C_k e^{-jk\omega_0 t_0}\right] e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \hat{C}_k e^{jk\omega_0 t}, \quad \text{where } \hat{C}_k = C_k e^{-jk\omega_0 t_0}$$

$|e^{-jk\omega_0 t}| = 1$

$\angle e^{-jk\omega_0 t} = -jk\omega_0 t$

$\therefore$ $|\hat{C}_k| = |C_k|$

$\angle \hat{C}_k = \angle C_k - jk\omega_0 t$

[4.6]

(a) $\int_0^{2\pi} \sin^2(t)\,dt$

$\Rightarrow \sin^2(t) = \frac{1+\cos 2t}{2}$

$$\int_0^{2\pi} \sin^2(t)\,dt = \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{\cos 2t}{2}\,dt$$

$$= \frac{1}{2}t\Big|_0^{2\pi} + \frac{1}{4}\sin 2t\Big|_0^{2\pi}$$

$$= \frac{2\pi}{2} + \frac{1}{4}(\sin 4\pi - 0)$$

$$= \pi$$

(b) $\int_0^{2\pi} \sin^2(2t)\,dt$

$\Rightarrow \sin^2(2t) = \frac{1+\cos 4t}{2}$

$$\int_0^{2\pi} \sin^2(2t)\,dt = \int_0^{2\pi} \frac{1+\cos 4t}{2}\,dt$$

$$= \frac{1}{2}t\Big|_0^{2\pi} + \frac{1}{8}(\sin 8\pi - 0)$$

$$= \pi$$

(c) $\int_0^{2\pi} \sin(t) \sin(2t)\, dt$.

$\Rightarrow \sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$.

$$\int_0^{2\pi} \sin(t) \sin(2t)\, dt = \frac{1}{2}\int_0^{2\pi} \cos(t) - \cos(3t)\, dt$$

$$= \frac{1}{2} \sin t \Big|_0^{2\pi} - \frac{1}{6} \sin 3t \Big|_0^{2\pi}$$

$$= \frac{1}{2}(0-0) - 1/6\,(0-0).$$

$$= 0.$$

(d) $e^{jxt} = \cos xt + j\sin xt$

$e^{jyt} = \cos yt + j\sin yt$

For orthogonality $\int_0^{2\pi} e^{jx} e^{jy}\, dt = 0$.

subsitute the above identity and simplify you

will get results in form of c to prove orthogonality

[4.17] Find m for the signals shown in the following figures:

From Section 4.4, property # 6

If the n^{th} derivative of $x(t)$ is the first derivative that contains discontinuity, then Fourier coefficients approach zero as $1/k^{n+1}$, prodvide all derivatives through n^{th} satisfy the Dirichlet conditions

$\therefore \; 1/k^m = 1/k^{n+1}$

$\Rightarrow m = n+1$

(a) For triangular wave, the first derivative $\frac{dx(t)}{dt}$

will have a discontinuity

$\therefore \; m = 2$

(b) For square wave $x(t)$ is discontinuous.

Also $\frac{d\,x(t)}{dt}$ is discontinuous

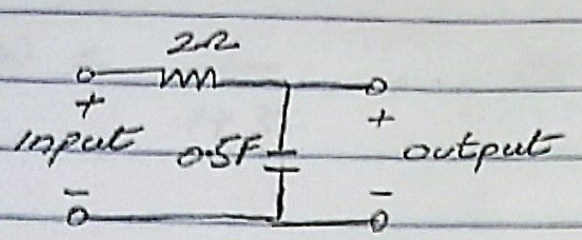
$\therefore \; m = 1$

(c) For sawtooth same as for part (b)

(d) $x_1(t)$ is continuous, $\frac{d\,x_1(t)}{dt}$ will be discontinuous

$m = 2$.

[4.26]



Square wave is applied at the input with $T_o = 2\pi$ s

$X_o = 10V$

(a) From the table 4.3, coefficients of square wave are

$C_o = 0$

$C_K = \frac{-j2X_o}{\pi K}$, $K =$ odd

$C_K = 0$, $K =$ even

The first three non-zero harmonics are C_1, C_3, C_5

$C_1 = \frac{-j20}{\pi}$, $C_3 = \frac{-j20}{3\pi}$, $C_5 = \frac{-j20}{5\pi}$

To calculate output we need the the Transfer Function of the system as $\boxed{C_{out} = C_{in} H(j\omega)}$

$H(j\omega) = \frac{1/j\omega C}{1/j\omega C + R}$

$= \frac{1}{1 + j\omega RC}$

For $T_o = 2\pi$ s

$\omega_o = 2\pi/2\pi = 1$ rad/s

$H(j\omega_o) = \frac{1}{1 + j(1)(2)(1/2)} = \frac{1}{1+j}$

$H_3(j3\omega_0) = \frac{1}{1+3j}$, $H_5(j5\omega_0) = \frac{1}{1+5j}$

$C_{o1} = \frac{-j20/\pi}{1+j}$ simplify to phasor form for Mag/Phase $\Leftrightarrow 4.5\angle -135°$

$C_{o3} = \frac{-j20/3\pi}{1+3j} \Leftrightarrow 0.67\angle -162°$

$C_{o5} = \frac{-j20/5\pi}{1+5j} \Leftrightarrow 0.25\angle -169°$

(c) All the results will remain the same, except for an added DC Value as $C_0 \neq 0$

$C_0 = 20$

$H_0(0) = 1$

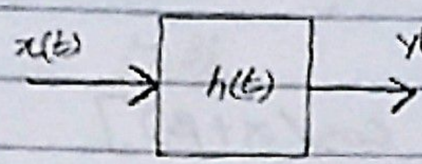
$C_{oo} = 20$

(d) Yes. Explain the transfer function

(e) $\omega_0 = 2$ rad/s , the transfer function becomes $\frac{1}{1+2j}$

This will affect the outputs harmonics but not the DC otwo

[4.30] $h(t) = e^{-\alpha t} u(t)$



(a) $H(s) = \int_{-\infty}^{\infty} e^{-\alpha t} u(t)\, e^{-st}\, dt$

$= \int_{0}^{\infty} e^{-(s+\alpha)t}\, dt$

$= \dfrac{1}{s+\alpha}$

For stable BIBO $\alpha > 0$

(b) $x(t) = \sin t + \cos(3t)$

$= \dfrac{1}{2j}\left(e^{jt} - e^{-jt}\right) + \dfrac{1}{2}\left(e^{3jt} + e^{-3jt}\right)$

$= \dfrac{e^{jt}}{2j} - \dfrac{e^{-jt}}{2j} + \dfrac{e^{3jt}}{2} + \dfrac{e^{-3jt}}{2}$

we know H(s) & each indivual component above in x(t) will generate a stable y(t) when multiplied with H(s)

$\therefore\ y(t) = \dfrac{e^{jt}}{2j(\alpha+j)} - \dfrac{e^{-jt}}{2j(\alpha-j)} + \dfrac{e^{3jt}}{2(\alpha+3j)} + \dfrac{e^{-3jt}}{2(\alpha-3j)}$

[4.31] $h(t) = \alpha e^{-\alpha t} u(t), \quad \alpha > 0$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} \alpha e^{-(s+\alpha)t}\, dt$$

$$= \frac{\alpha}{s+\alpha}$$

(a) $x(t) = \sin^2(2t)$

$$= \tfrac{1}{2}(1-\cos(4t))$$

$$= \tfrac{1}{2}\left(1 - \frac{e^{4jt}}{2} - \frac{e^{-4jt}}{2}\right)$$

as before $H(0) = 1, \quad H(4j) = \frac{\alpha}{4j+\alpha}, \quad H(-4j) = \frac{\alpha}{-4j+\alpha}$

$$y(t) = \frac{1}{2} - \frac{e^{4jt}}{4}\left(\frac{\alpha}{4j+\alpha}\right) - \frac{e^{-4jt}}{4}\left(\frac{\alpha}{-4j+\alpha}\right)$$

(b) $x(t) = 1 + \cos t + \cos(8t)$

$$= 1 + \frac{e^{jt}}{2} + \frac{e^{-jt}}{2} + \frac{e^{8jt}}{2} + \frac{e^{-8jt}}{2}$$

$H(0) = 1, \quad H(j) = \frac{\alpha}{\alpha - j}, \quad H(-j) = \frac{\alpha}{\alpha - j}, \quad H(8j) = \frac{\alpha}{\alpha + 8j}, \quad H(8j) = \frac{\alpha}{\alpha - 8j}$

$$y(t) = 1 + \frac{\alpha e^{jt}}{2(\alpha+j)} + \frac{\alpha e^{-jt}}{2(\alpha-j)} + \frac{\alpha e^{8jt}}{2(\alpha+8j)} + \frac{\alpha e^{-8jt}}{2(\alpha-8j)}$$

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